Sixteenth Annual Iowa Collegiate Mathematics Competition

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Your solutions should be clearly written arguments. Merely stating an answer without any justication will receive little credit.

1 The Mad Veterinarian

In popular culture, many of us are familiar with the stereotype of the mad scientist. In this case, a mad veterinarian invents an animal transmogrifying machine. The machine can transmogrify:

- Two cats into one cat, or vice-versa
- One cat and one dog into one dog, or vice-versa
- Two dogs into one cat, or vice-versa

Beginning with three cats and one dog, is it possible to end up with

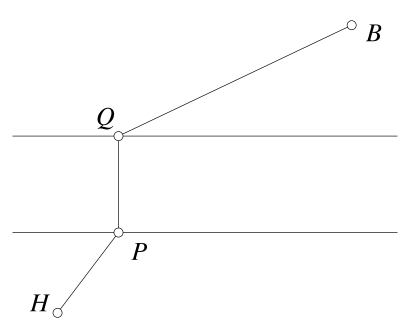
(a) one dog and no cats?

(b) one cat and no dogs?

Be sure to justify your answers.

2 The shortest walk

A farmer lives in a farmhouse H on one side of a stream bounded by two parallel lines. He often has to walk to his barn B on the other side of the stream. Since he is tired of getting wet, he wants to build a bridge PQ perpendicular to the stream, with P on the same side of the stream as H. He also wants the total walking distance $HP+PQ+QB$ to be as short as possible. How should he determine where to place the bridge?



3 Odd divisor sums

What positive integers n have a divisor sum that is odd? For instance, the sum of the divisors of 6 is even since $1+2+3+6=12$, and the sum of the divisors of 9 is odd since $1+3+9=13$.

4 The closest playground

Four families, A, B, C, and D, live in houses at the vertices of a convex quadrilateral.

They decide to put their money together and build a new playground P, located at the point where the sum of the distances $PA+PB+PC+PD$ is as small as possible.

Describe how to determine the location of P and prove that the sum of the distances is minimal.

5 Three-cycles

Find a *real-valued* function f such that the third derivative $f'''(x)=f(x)$ for all x, and f is not a constant multiple of e^x. Express f in terms of elementary functions (that is, not a power series or limit or other such representation).

6 Four cups, one table

Four empty cups, which can be flipped either up or down, sit at the four compass points on a table. On each round of the game, player 1 chooses which cup(s) to flip, such as "flip the north and west cups". Then player 2 first flips those cups and then rotates the table by any multiple of 90 degrees (including 0) to reposition them. Player 1 wins as soon as all cups are up.

Unfortunately for player 1, the rules require that:

- Player 1 must write down a finite list of instructions for every turn of the game before the game begins.
- Player 2 may look at that list before choosing the initial position of the cups.

Despite this, it is possible for player 1 to guarantee victory! How long is the shortest list of winning instructions for player 1? Give an example of such a list and explain how you can be sure player 1 will win.

7 A minimal area

A smooth function $f(x)$ has $f''(x) > 0$ for all x in $[0,1]$.

For each point a in $[0,1]$, draw the tangent line to $y = f(x)$ at the point where $x = a$. Let $A(a)$ be the area bounded by the curve $y = f(x)$, the tangent line at a, $x = 0$, and $x = 1$.

For what value of a is the area minimized?

8 Only somewhat deranged

How many rearrangements of the string of letters *aabcde* have exactly two letters in their original places? The two a's are indistinguishable, so an a in either the first or second position is considered to be in its original place.

9 Coins in Twoland

In Twoland, the government mints one type of coin for each power of 2, and the base unit of their currency is called the tooie. They also have some unusual laws that require all purchases to be made with exact change, and at most two of each type of coin to be used for any given purchase. For example, to pay 6 tooies, it would be legal to pay with $4+2$ or $4+1+1$ or $2+2+1+1$ but not $2+2+2$ nor 8. It is considered the same way to pay with $4+1+1$ or $1+4+1$ or any other rearrangement of the same coins in a different order; only the collection of coins matters. So there are three ways to pay 6 tooies.

(a) How many ways are there to pay for an item costing 20 tooies?

(b) How many ways are there to pay for an item costing 1000 tooies?

10 Square of sum is sum of cubes

Prove that, for any natural number N, if $d_1, d_2, \ldots, d_k$ are the divisors of N, and $n_1, n_2, \ldots, n_k$ are how many divisors each of $d_1, d_2, \ldots, d_k$ have, then

$$(n_1 + n_2 + \cdots + n_k)^2 = n_1^3 + n_2^3 + \cdots + n_k^3.$$