Fifteenth Annual Iowa Collegiate Mathematics Competition
Iowa State University February 28, 2009

Problems by
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The problems are listed (roughly) in order of difficulty. Each solution requires a proof or justification. Answers only are not enough. Calculators are allowed but certainly not required.

1. Product 1, Sum 0

The product of 30 integers is one. Can their sum be zero?

2. As Easy as 3-4-5

A circle of radius r is inscribed in a right triangle with leg 4r.

Prove that the triangle is a 3-4-5 right triangle.

3. Speaking of Pythagoras

Prove that the inradius of any Pythagorean right triangle (a right triangle with integer side lengths) has integer length.

4. Equal Integrals

Let F be a polynomial function of degree $2n$,

let G be a polynomial function of degree $2n+1$, and suppose that,

for a and for some $d > 0$, $F(a+id) = G(a+id)$ for $i = 0, 1, 2, \ldots, 2n$.

Let $b=a+(2n)d$. Prove that $\int_a^b F(x)\,dx = \int_a^b G(x)\,dx$

5. Φ Fun

Let $\Phi = \dfrac{1+\sqrt{5}}{2}$. Given positive real numbers X and Y with $X > \Phi Y$.

Prove that $\dfrac{X+Y}{X}$ is closer to Φ than $\dfrac{X}{Y}$ is.

6. *N* up, *N* down

Choose N elements of $\{1, 2, 3, \ldots, 2N\}$ and arrange them in increasing order.

Arrange the remaining N elements in decreasing order. Let D_i be the absolute value of the difference of the ith elements in each arrangement. Prove that

$$D_1 + D_2 + \ldots + D_N = N^2$$

7. Coin Tossing

Al tosses a fair coin n times and Babs tosses a fair coin $n+k$ times.

Prove that the probability that Al tosses at least as many heads as Babs tosses is

$$\frac{{}_{2n+k}C_0 + {}_{2n+k}C_1 + \ldots + {}_{2n+k}C_n}{2^{2n+k}}$$

8. Function Phenomenon

Let F and G be real valued functions defined on [0,1].

Prove that there exist a and b in [0,1] such that

$$| ab - F(a) - G(b) | \geq 1/4.$$

9. On the Fence Post

There are n posts, numbered 1 through n, arranged in a circle, and there are k colors of paint available. Prove that the number of different ways the posts can be painted so that adjacent posts have different colors is

$$P_n = (k-1)^n + (-1)^n (k-1)$$

10. Integer Sticks

A line segment with odd integer length $n = 2k+1$ is randomly cut into three pieces, each of integer length. What is the probability that the three pieces can be formed into a (non-degenerate) triangle?