Fourteenth Annual Iowa Collegiate Mathematics Competition

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1. The integer m is not a square of an integer. To see this we notice that the last digit of a square of an integer can only be: 0, 1, 4, 5, 6, or 9.

   Alternatively, write the integer m the form

   $$m = 123456\ldots 20070000 + 2008.$$

   Then m is of the form $16k + 8$, for some integer k. Hence the highest power of 2 that divides m is 3.

2. Let the point (x, y, z) be the vertex on the plane $x + y + z = 1$. Then the question asks for finding a maximum possible value of the product xyz.

   The geometric and arithmetic means inequality gives

   $$\sqrt[3]{xyz} \leq \frac{x+y+z}{3} \quad \text{so,} \quad xyz \leq \left(\frac{x+y+z}{3}\right)^3 \leq \frac{1}{27}.$$

   This maximum value can be achieved by taking $x = y = z = \frac{1}{3}$.

   Alternatively, we can write the volume of the box as function of two variables:

   $$V(x, y) = xy(1 - x - y), \quad \text{where } x \geq 0,\ y \geq 0, \text{ and } 0 \leq x \leq 1.$$

   Setting the partial derivatives equal to zero gives

   $$V_x(x, y) = y(1 - 2x - y) = 0 \quad \text{and} \quad V_y(x, y) = x(1 - 2y - x) = 0,$$

   and the only common solutions to these equations are

   $$(0, 0),\ (0, 1),\ (1, 0), \quad \text{and} \quad \left(\frac{1}{3}, \frac{1}{3}\right).$$

   The volume is zero at (0,0), (0,1), and (1,0), hence the maximum occurs at the point $(1/3, 1/3)$, giving the maximum value

   $$V\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}.$$

The problem can also be solved by applying the method of Lagrange multipliers. In this case, one obtains the equations:

$$yz = \lambda,\ xz = \lambda,\ xy = \lambda, \quad \text{and} \quad x + y + z = 1.$$

If at least one of x,y, or z is equal to 0, then the volume is also zero, so we can assume that they are positive. Hence the equations imply that

$$x = y = z = \frac{1}{3},$$

and the value for the maximum volume being $V = 1/27$.

3. When $x = 0$, the functional relationship implies that

$$f(y) = f(0 + y) = y + f(0),$$

so f is a linear function. By the divisibility condition we have

$$2+f(0) \mid 5+f(0) \quad \text{if and only if} \quad 2+f(0) \mid (5+f(0))-(2+f(0)) = 3,$$

so we must have $f(0) = 1$. Therefore $f(y) = y+1$ and $f(2008) = 2009$.

Alternatively, one can get a linear function by taking $y = -x$. This time the equation takes form $f(x) - x = f(0)$, and the rest of the solution is the same as above.

4. Take the derivative of each side to obtain, for every $x \geq 0$,

$$3(f(x))^2 f'(x) = x(f(x)^2, \quad \text{that is,} \quad (f(x))^2(3f'(x) - x) = 0.$$

Hence for all $x > 0$ we have either $f(x) = 0$ or $3f'(x) - x = 0$. Since $f(0)$ is expressed in terms of a definite integral from 0 to 0, therefore $f(0) = 0$. Because $f(x)$ is strictly increasing, it follows that $f(x) > 0$ for all $x > 0$. Thus $3f'(x) - x = 0$ for all $x > 0$, and

$$f'(x) = \frac{x}{3}, \quad \text{for all } x \geq 0.$$

It follows that $f(x) = x^2/6$, for all $x \geq 0$.

5. We need to find positive integers k and m, such that $m \geq 2$, and

$$(k+1) + (k+2) + \ldots + (k+m) = 2008.$$

The left hand side of this equation is a difference between the sum of all positive integers $\leq k+m$, and the sum of all positive integers $\leq k$, so it is equivalent to

$$\begin{aligned} 2008 &= \frac{(k+m)(k+m+1)}{2} - \frac{k(k+1)}{2} \\ &= \frac{1}{2}(k^2 + 2km + m^2 + k + 1 - k^2 - k) \\ &= \frac{1}{2}m(m+2k+1). \end{aligned}$$

Thus

$$m(m+2k+1) = 4016 = 16 \cdot 251.$$

The factors m and $m+2k+1$ are of different parity, and $m+2k+1 > m$. It follows that $m = 16$, and $m+2k+1 = 251$, giving $k = 117$. Therefore $2008 = 118 + 119 + \cdots + 133$.

6. Yes, the sets are equal. The following divisibility conditions are equivalent:

$$13 \mid a+5b \iff 13 \mid 16(a+5b) \iff 13 \mid 16(a+5b) - 13(a+6b) = 3a+2b.$$

7. Consider the function $g(x) = xf(x)$. Since $f(1) = 0$, it follows that $g(0) = g(1) = 0$, and thus the function g satisfies the assumptions of the Rolle's Theorem on the interval $[0,1]$. Therefore $g'(c) = 0$ at some number $0 < c < 1$. So

$$0 = g'(c) = f(c) + cf'(c) \quad \text{and} \quad \frac{f(c)}{c} = -f'(c).$$

8. Since $(1+ri)^3 = 1 + 3ri - 3r^2 - r^3 i$, then the condition of the problem implies that $1 - 3r^2 = 3r - r^3$, or equivalently

$$0 = r^3 - 3r^2 - 3r + 1 = (r+1)(r^2 - 4r + 1).$$

Hence the roots are

$$r_1 = -1, \quad r_2 = 2 - \sqrt{3}, \quad \text{and} \quad r_3 = 2 + \sqrt{3}.$$

Therefore

$$r_1^2 + r_2^2 + r_3^2 = 15.$$

Alternatively, writing

$$r^3 - 3r^2 - 3r + 1 = (r - r_1)(r - r_2)(r - r_3),$$

and comparing coefficients implies that

$$r_1 + r_2 + r_3 = 3 \quad \text{and} \quad r_1 r_2 + r_1 r_3 + r_2 r_3 = -3.$$

So

$$r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1 r_2 + r_1 r_3 + r_2 r_3) = 3^2 - 2 \cdot (-3) = 15.$$

9. The given equation is equivalent to

$$AB - A - B + I = I, \quad \text{or to} \quad (A - I)(B - I) = I.$$

Since the matrices have integer entries, the determinants of $A - I$ and $B - I$ are integers, and the last equation implies that $\det(A - I) = \det(B - I) = \pm 1$.

Both cases are possible: if $A = B = O$, then $\det(A - I) = \det(-I) = -1$, and if $A = B = 2I$, then $\det(A - I) = \det(I) = 1$.

10. Let

$$g(x) = \frac{f'(x)}{f(x)}.$$

The inequality

$$2(f'(x))^2 \geq (f(x))^2 + (f''(x))^2 \geq 2f(x)f''(x),$$

implies that

$$g'(x) \leq 0,$$

and therefore $g(x)$ is non-increasing on the interval [0,1]. Hence

$$\ln f(1) = \int_0^1 \frac{f'(x)}{f(x)}\, dx \leq g(0) \cdot 1 = 1,$$

which implies that $f(1) \leq e$. The function $f(x) = e^x$ satisfies the conditions of the problem, with $f(1) = e$.