Fourteenth Annual Iowa Collegiate Mathematics Competition
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To receive full credit, all problems require complete justification.

1. The integers from 1 to 2008 are written consecutively in a row to form a new integer
$$m = 123456\ldots 20072008.$$
Determine whether m is a square of an integer.

2. A rectangular box lies in the first octant, with all its edges parallel to the coordinate axes. One vertex of the box is in the origin, and the diagonally opposite vertex is on the plane $x + y + z = 1$. Find the maximum possible volume of the box.

3. Let f be a real function satisfying $f(x)+y = f(x+y)$ for all real x and y. Assume that $f(0)$ is a positive integer, and that $f(2) \mid f(5)$. Find $f(2008)$.
Note: For integers m and n, the symbol $m \mid n$ means that m divides n.

4. Let $f : [0, \infty) \to R$ be a continuous, strictly increasing function, such that
$$(f(x))^3 = \int_0^x t(f(t))^2\,dt$$
for every $x \geq 0$. Show that for every $x \geq 0$ we have $f(x) = \dfrac{x^2}{6}$.

5. Express 2008 as a sum of consecutive positive integers.

6. Let
$A = \{(a, b) : a, b \text{ in } \mathbb{Z} \text{ and } (a + 5b) \text{ is an integer multiple of } 13\}$,
$B = \{(c, d) : c, d \text{ in } \mathbb{Z} \text{ and } (3c + 2d) \text{ is an integer multiple of } 13\}$.
Decide whether $A = B$.

7. Let f be a continuous function on $[0,1]$, differentiable on $(0,1)$, and such that $f(1)=0$. Show that for some $c \in (0,1)$,

$$\frac{f(c)}{c} = -f'(c).$$

8. Find the sum of squares of all real numbers r, for which the expression $(1+ri)^3$ is of the form $s(1+i)$ for some real number s, where $i^2=-1$.

9. Let A and B be 3×3 matrices with integer entries, such that

$$AB = A + B.$$

   Find all possible values of $\det(A-I)$.
   Note: The symbol I represents the 3×3 identity matrix.

10. Let f be a real, positive and continuous function on the interval $[0,1]$, twice differentiable on $(0,1)$, and such that $2(f')^2 \geq f^2 + (f'')^2$ on $(0,1)$. Suppose also that $f(0)=f'(0)=1$. Find the maximum possible value for $f(1)$.