(1) Let \( a > 0 \), and define the following function:

\[
f(x) = \frac{\sqrt{a^4 x} - a\sqrt{a^2 x}}{a - \sqrt{ax^2}}.
\]

- Calculate these limits:

\[
\lim_{x \to 0^+} f(x) =
\lim_{x \to a} f(x) =
\lim_{x \to +\infty} f(x) =
\]

- Find the maximum value of \( f(x) \) on its domain.

**Solution:** The function

\[
f(x) = \frac{a^{3/2}x^{1/3}(x^{1/6} - a^{1/6})}{a^{1/4}(a^{3/4} - x^{3/4})}
\]

is continuous on \([0, a)\), with the \( x \to 0^+ \) limit equal to 0.

For the \( x \to a \) limit, the \( ^0 \) form of L'Hôpital's Rule applies.

\[
\lim_{x \to a} \sqrt{a^2x} - a\sqrt{a^2x} = \lim_{x \to a} \frac{a^{3/2}x^{1/2} - a^{5/3}x^{1/3}}{a - a^{1/4}x^{3/4}}
\]

\[
(LHR) = \lim_{x \to a} \frac{a^{3/2}1_2 x^{-1/2} - a^{5/3}1_3 x^{-2/3}}{0 - a^{1/4}_3 x^{-1/4}}
\]

\[
= \frac{\frac{1}{2}a - \frac{1}{3}a}{-\frac{3}{4}}
\]

\[
= \frac{2a}{9}
\]

For the \( x \to +\infty \) limit, LHR could be used again, but it is easier to notice \( f \) satisfies \( |f(x)| < Cx^{-1/4} \) for large \( x \), so there is a horizontal asymptote \( f(x) \to 0 \) as \( x \to \infty \).

From the above expression, \( f(x) < 0 \) for all \( x \in (0, a) \cup (a, \infty) \), so the maximum value is \( f(0) = 0 \).

**Comment:** You may try to find critical points for the second part, but this is a difficult calculation and a waste of time.

(2) Let \( f \) be a function with domain \((0, \infty)\) satisfying:

- \( f(x) = f(x^2) \) for all \( x > 0 \)
- \( \lim_{x \to 0^+} f(x) = \lim_{x \to +\infty} f(x) = f(1) \)

Show that \( f(x) \) is a constant function on \((0, \infty)\).

**Solution:** For integer \( k \geq 1 \), \( f(x^{2k}) = f((x^{2k-1})^2) = f(x^{(2k-1)}) \), so by induction, \( f(x^{2k}) = f(x) \) for all integer \( k \geq 0 \). Given \( \epsilon > 0 \), there is some \( \delta \in (0, 1) \) and some \( N \in (1, \infty) \) so that if \( 0 < t < \delta \) or \( t > N \), then \( |f(t) - f(1)| < \epsilon \). If \( 0 < x < 1 \), then there is some integer \( k \) such that
$k > \log_2(\ln(\delta)/\ln(x))$, which is equivalent to $0 < x^{(2^k)} < \delta$, and if $x > 1$, then there is some integer $k$ such that $k > \log_2(\ln(N)/\ln(x))$, which is equivalent to $N < x^{(2^k)}$, so in either case, $|f(x) - f(1)| = |f(x^{(2^k)}) - f(1)| < \varepsilon$. Since $\varepsilon$ was arbitrary, $f(x) = f(1)$.

**Comment:** You may try using more informal limit arguments, but at the risk of taking some unjustified steps.

(3) Let $V$ be a corner of a right-angled box and let $x, y, z$ be the angles formed by the long diagonal and the face diagonals starting at $V$. For

$$A = \begin{bmatrix} \sin x & \sin y & \sin z \\ \sin z & \sin x & \sin y \\ \sin y & \sin z & \sin x \end{bmatrix}$$

show that $|\det(A)| \leq 1$.

**Solution:** From $\sin = \frac{\text{opp}}{\text{hyp}}$, $\sin x = a/d$, $\sin y = b/d$, and $\sin z = c/d$, where $a, b, c$ are the side lengths of the box and $d$ is the long diagonal length.

$$\det(A) = \frac{1}{d^3} \det \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

The absolute value of

$$\det \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

is the volume of a parallelepiped with side lengths all equal to $d$. By the scalar triple product formula, such a volume is maximized when the parallelepiped has all right angles, so it is a cube with volume $d^3$. The claimed inequality follows.

**Comment:** Is there some less geometric approach, maybe an obscure inequality comparing $\det = a^3 + b^3 + c^3 - 3abc$ to $d^3 = (a^2 + b^2 + c^2)^{3/2}$? The authors would be interested to know.

(4) Let $f(t)$ be a real valued integrable function on $[0,1]$, so that both sides of the following equation are continuous functions of $x$:

$$2x - 1 = \int_0^x f(t)dt.$$

Prove that if $f(t) \leq 1$ for $0 \leq t \leq 1$, then there exists a unique solution $x \in [0,1]$ of the equation.
Solution: Let \( F(x) \) be the function \( 2x - 1 - \int_0^x f(t)dt \), which is continuous on \([0, 1]\) and satisfies \( F(0) = -1 \) and \( F(1) = 1 - \int_0^1 f(t)dt \geq 1 - \int_0^1 1dt = 0 \). By the Intermediate Value Theorem, \( F(x) = 0 \) has at least one solution \( x \in [0, 1] \). This solution is unique because \( F(x) \) is increasing on \([0, 1]\): for \( 0 \leq a < b \leq 1 \),
\[
F(b) - F(a) = 2(b - a) - \int_a^b f(t)dt \geq 2(b - a) - 1(b - a) = b - a > 0
\]

Comment: If \( f(t) \) were continuous, then \( F(x) \) could be proved increasing using the Fundamental Theorem of Calculus: \( F'(x) = 2 - f(x) \geq 1 \). However, the problem specifically omits this hypothesis.

(5) Let \( ABCD \) be a rectangle. The bisector of the angle \( ACB \) intersects \( AB \) at point \( M \) and divides the rectangle \( ABCD \) into two regions: the triangle \( MBC \) with area \( s \) and the convex quadrilateral \( MADC \) with area \( t \).

- Determine the dimensions of the rectangle \( ABCD \) in terms of \( s \) and \( t \).
- If \( t = 4s \), what is the ratio \( AB/BC \)?

Solution: Let \( AB = b, BC = h, AM = y, MB = x \), and let \( \theta \) be half the angle \( ABC \), and let \( \alpha \) be the angle \( BMC \). By the Law of Sines,
\[
\sin \theta = \frac{\sin(\pi - \alpha)}{AC}, \quad \sin \alpha = \frac{\sin \theta}{x} \implies \frac{\sin \alpha}{\sin \theta} = \frac{h}{x} = \frac{AC}{y} = \frac{\sqrt{b^2 + h^2}}{y}
\]

We have the following system of polynomial equations.
\[
\begin{align*}
x + y &= b \\
\frac{1}{2}xh &= s \\
hh &= s + t \\
x^2(b^2 + h^2) &= h^2y^2
\end{align*}
\]

Eliminating \( y \) first gives:
\[
x^2(b^2 + h^2) = h^2(b - x)^2 \implies x^2b = h^2b - 2h^2x
\]

Multiplying both sides by \( h^3 \) gives:
\[
\begin{align*}
x^2bh^3 &= h^4(hb - 2hx) \\
(2s)^2(s + t) &= h^4(s + t - 2(2s)) \\
h &= \left(\frac{4s^2(s + t)}{t - 3s}\right)^{1/4} \\
b &= \frac{s + t}{h} = \frac{(s + t)^{3/4}(t - 3s)^{1/4}}{\sqrt{2s}}
\end{align*}
\]

The \( b/h \) ratio can be computed directly for \( t = 4s \), or as:
\[
\frac{b}{h} = \frac{bh}{h^2} = \frac{s + 4s}{\left(\frac{4s^2(s + 4s)}{4s - 3s}\right)^{1/2}} = \frac{5s}{\sqrt{20s^2}} = \frac{\sqrt{5}}{2}
\]

Comment: The equality of ratios \( \frac{b}{h} = \frac{AC}{y} \) from the first step is also known as the “bisector theorem” for triangles.
(6) In a badly overcrowded pre-school, every child is either left-handed or right-handed, either blue-eyed or brown-eyed, and either a boy or a girl. Exactly half of the children are girls, exactly half of the children are left-handed and exactly one fourth of the children are both. There are twenty-six children who are brown-eyed. Nine of those twenty-six are right-handed boys. Two children are right-handed boys with blue eyes. Thirteen children are both left-handed and brown-eyed. Five of these thirteen are girls.

- How many students does the pre-school have?
- How many girls are right-handed and blue-eyed?

**Solution:** There are 8 types of students with the following populations:

- # RH BL boy = 2
- # RH BR boy = 9
- # LH BR girl = 5
- # LH BR boy = 13
- # LH BL girl = x
- # RH BL girl = y
- # LH BL boy = z

From equal numbers of boys and girls, \( x + y + 9 = 19 + z \). From equal numbers of LH and RH, \( x + z + 13 = y + 15 \). From one fourth LH girls, \( 4(x + 5) = x + y + z + 28 \). This is a system of three linear equations in three unknowns. Standard solution methods give the unique answer \( x = 6, y = 7, \) and \( z = 3 \), so the total population is \( x + y + z + 28 = 44 \), with 7 RH BL girls.

**Comment:** Drawing a Venn diagram may be helpful.

(7) Let \( n > 1 \) be an integer. Let \( (G, \cdot) \) be a group, with an identity element \( e \) and an element \( a \in G \) with \( a \neq e \) and \( a^n = e \). Let \( (H, \ast) \) be a group, let \( f : G \to H \) be an arbitrary function, and then define \( F : G \to H \) by:

\[
F(x) = f(x) \ast f(a \cdot x) \ast f(a^2 \cdot x) \ast \ldots \ast f(a^{n-1} \cdot x)
\]

- Show that if \( f(G) \) is a subset of some Abelian subgroup of \( H \), then \( F \) is not a one-to-one function.
- Let \( (H, \ast) \) be the symmetric group \( (S_3, \circ) \) (the six-element group of permutations of three objects). Give an example of \( (G, \cdot), n, a \) as above, and a function \( f : G \to H \), so that the expression \( F \) is a one-to-one function.

**Solution:** For the first part,

\[
F(e) = f(e) \ast f(a \cdot e) \ast f(a^2 \cdot e) \ast \ldots \ast f(a^{n-1} \cdot e)
\]

\[
= f(e) \ast f(a) \ast f(a^2) \ast \ldots \ast f(a^{n-1})
\]
\[ F(a) = f(a) \ast f(a^2) \ast f(a^3) \ast \ldots \ast f(a^{n-2} \cdot a) \ast f(a^{n-1} \cdot a) \]
\[ = f(a) \ast f(a^2) \ast f(a^3) \ast \ldots \ast f(a^{n-2}) \ast f(a^{n-1}) \ast f(e) = F(e) \]

using the property that \( f(e) \) commutes with other \( f(g) \) at the last step. By the assumption that \( a \neq e \), \( F \) is not one-to-one.

For the second part, there are lots of examples. (A correct answer must have explicit examples of \( G, n, a, \) and \( f \).) A simple one is to let \( G \) be a two element group \( \{e, a\} \), so \( n = 2 \), and to define \( f : G \rightarrow S_3 \) by \( f(e) = (12) \) and \( f(a) = (23) \), or any other pair of non-commuting elements in \( S_3 \). Then \( F(e) = f(e) \ast f(a) = (12) \circ (23) = (123) \) and \( F(a) = f(a) \ast f(e) = (23) \circ (12) = (132) \), so \( F \) is one-to-one.

(8) Determine whether the following sum of real cube roots is rational or irrational:
\[ \sqrt[3]{6} + \frac{\sqrt[3]{847}}{27} + \sqrt[3]{6 - \frac{\sqrt[3]{847}}{27}} \]

**Solution:** Let \( x \) be the number; then a short calculation with convenient cancellations shows \( x \) satisfies \( x^3 = 12 + 5x \). The only real root of \( x^3 - 5x - 12 = (x - 3)(x^2 + 3x + 4) \) is \( x = 3 \).

**Comment:** This is similar to problem #3 from the 1969 ICMC.