Define a sequence \((s_n)\) recursively as follows: Let \(s_1 = 1\) and for \(n \geq 1\), let \(s_{n+1} = \sqrt{1 + s_n}\). Prove that \((s_n)\) converges, and then find the limit.

Show work to be graded below, and use the reverse side of the page to continue if necessary.
Let $C$ be a non-empty collection (possibly infinite) of compact subsets of $\mathbb{R}$.

(1) Prove that $K = \bigcap_{C \in C} C$ is a compact set.

(2) Give an example that illustrates that the union of a family of compact sets need not be compact.

Show work to be graded below, and use the reverse side of the page to continue if necessary.
Assume $A$ and $B$ are two sets with $m$ and $n$ elements, respectively.

(1) How many one-to-one functions are there from $A$ and $B$?

(2) How many one-to-one and onto functions are there from $A$ to $B$?

Show work to be graded below, and use the reverse side of the page to continue if necessary.
Let $p$ and $q$ be distinct prime numbers. Find the number of generators of the group $\mathbb{Z}_{pq}$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.
Let $G$ be a group and $H$ a subgroup of $G$ with index $(G : H) = 2$. Prove that $H$ is a normal subgroup of $G$.

Show work to be graded below, and use the reverse side of the page to continue if necessary.
The Fibonacci numbers are defined as
\[ f_1 = f_2 = 1 \]
and
\[ f_{n+1} = f_n + f_{n-1} \]
for \( n \geq 3 \).

(1) List \( f_1, f_2, \ldots, f_7 \).
(2) Illustrate, using the list from (a), that \( f_{2n+1} = f_{n+1}^2 + f_n^2 \) for \( n = 1, 2, 3 \).
(3) Prove that \( f_{2n+1} = f_{n+1}^2 + f_n^2 \) for all \( n \in \mathbb{N} \).

Show work to be graded below, and use the reverse side of the page to continue if necessary.
Let \( a, b, m, M \) be real numbers with \( 0 < m \leq a \leq b \leq M \), prove that

\[
\frac{2\sqrt{mM}}{m + M} \leq \frac{2\sqrt{ab}}{a + b}
\]

Show work to be graded below, and use the reverse side of the page to continue if necessary.
A soccer ball is stitched together using white hexagons and black pentagons. Each pentagon borders five hexagons. Each hexagon borders three other hexagons and three pentagons. Each vertex is of valence 3 (meaning that at each corner of a hexagon or pentagon, exactly three hexagons or pentagons meet). How many hexagons and how many pentagons are needed to make a soccer ball? **Hint:** Euler’s Polyhedron Formula states that $V - E + F = 2$, where $V$ is the number of vertices, $E$ is the number of edges (i.e., the line adjoining two vertices) and $F$ is the number of faces (hexagons or pentagons).

Show work to be graded below, and use the reverse side of the page to continue if necessary.