

ISMAA 2023 Abstracts
College of DuPage
March 31 & April 1, 2023

Pre-conference Workshop

Manda Riehl, Rose-Hulman Institute of Technology, *Metacognition: Teaching Students to Think about Learning*

Metacognition involves our knowledge and regulation of our learning processes. Everyone uses metacognition to some degree, but all students (and faculty) have room to further develop their metacognitive skills. Many students rely on memorization and rehearsal as their main approaches to learning, when they could benefit from organizational approaches associated with lasting learning. This workshop will focus on understanding metacognition, tools faculty can use in their classrooms to improve student metacognition, and giving feedback to students on their metacognition. Faculty will leave with a concrete activity tailored to their course, students, and teaching style.

Plenary Talks

Manda Riehl, Rose-Hulman Institute of Technology, *Discrete Mathematics Applied to Biology*
Discrete mathematics is a useful field for computer scientists and electrical engineers, covering essential topics including logic, Boolean algebra, theory of computation, recursion, and many others. But recently the tools of discrete mathematics have brought about advances in the biological sciences, particularly in modeling biological systems and processes. We'll explore some applications of discrete mathematics to topics such as gene regulatory networks, predicting evolution, and RNA secondary structures.

Hortensia Soto, Colorado State University, *Intentional Integration of Embodiment forms to teach the FHT*

In this case study, we explored how a mathematics education researcher integrated embodiment beyond gesture as she developed an experiential foundation for the Fundamental Homomorphism Theorem (FHT) in a first semester abstract algebra course. We found that this instructor intentionally used embodiment to support student contributions and to reduce levels of abstraction for the formal definitions, theorems, and proofs. In addition, she encouraged students to interact with physical materials and simulate the mathematics with their bodies. Simulations opened communication lines between the instructor and students, who were not fluent in formal language. The instructor's simultaneous use of various forms of embodiment primed students for the formalism and symbolism, highlighted and disambiguated students' referents, amplified students' contributions to develop fluency, and linked students' body form catchments to reinforce the FHT. Our results offer practical implications for teaching by illustrating examples of how embodiment can be implemented into an abstract algebra classroom.

Bob Cappetta, Florida Southwestern State College, *Active Learning in Calculus*

The leaders of MAA and other mathematics professional societies have concluded that effective active learning is essential to improving student performance and confidence. This presentation will examine the evidence that supports this claim, and it will share suggestions and sample problems for implementation. Additionally, participants will have opportunities to discuss and share strategies and concerns regarding active learning in the calculus curriculum.

Chris Polly, Fermi National Accelerator Laboratory, *Everyday Math in the Pursuit of Big Science: A Muon $g-2$ Case Study*

The Muon $g-2$ experiment at Fermilab utilizes an intense beam of subatomic particles, called muons, to search for telltale signs of new particles or yet undiscovered fundamental forces of nature. Published in 2021, initial results from the experiment confirmed a long-standing mystery first observed at Brookhaven National Laboratory over 20 years ago. The results seem to indicate a crack may be forming in the Standard Model of particle physics, the bedrock of physics at the most fundamental level. The experiment's evolution from proposal to the front page of the New York Times will be told with an emphasis on how the mathematics taught to our students gets put to everyday practical use in experimental searches that have the potential to shake our core understanding of the universe.

Contributed Talks—Friday

Joe Stickles, Millikin University, *A Primer on Graphs Associated with Algebraic Structures*

In 1988, Istvan Beck introduced a simple graph associated with a commutative ring. The vertices of this graph are all the elements of the ring, and two distinct vertices are connected with an edge if the product of the two vertices is zero in the ring. In 1998, the vertex set was modified to include just the nonzero zero-divisors of the ring, and this is now accepted as the zero-divisor graph of the ring. Since this paper, literally hundreds of articles have appeared that investigate this and other graphs associated with algebraic structures, many of which have been authored, at least in part, by undergraduates. In this talk, we will introduce some of these graphs, discussing their fundamental properties and what they might tell us about the underlying algebraic structure. We will also discuss some recent results concerning domination in zero-divisor graphs.

Ellen Ziliak, Benedictine University, *I'm Puzzled: Using Algebra and Grobner Basis to Solve Problems*

In Linear Algebra one of the first topics covered is Gaussian Elimination for a linear system of equations. Once students learn this algorithm the course focuses on numerous applications of this algorithm. However, when we get to Abstract Algebra and study the theory of polynomial rings and finite fields there is an equally useful algorithm the Buchberger's Algorithm which produces a Grobner Basis for any system of polynomial equations. This algorithm is equivalent to Gaussian Elimination if the system is linear, equivalent to the Euclidean Algorithm if our system consists of univariate polynomials, and is a natural extension of the division algorithm. In this presentation I will discuss how I include Grobner Basis in my Abstract Algebra classes, and many of the interesting puzzles that my students have used this algorithm to solve.

Cara Sulyok, Lewis University, *Undergraduate Research Aimed at Solving Clostridioides difficile: Mathematical Models of Transmission and Control in Healthcare Settings*

Clostridioides difficile (*C. difficile*) is the leading cause of infectious diarrhea and the most frequently identified healthcare-acquired infection in United States hospitals. *C. difficile* is typically contracted after antibiotic use, when healthy gut microbiota that prevent colonization is compromised. Colonized patients, both symptomatic and asymptomatic, shed *C. difficile* endospores that can survive for long periods on surfaces outside the host and are resistant to many commonly-used disinfectants. Transmission pathways can include contact with endospores on fomites, objects likely to carry infection.

This talk will focus on various mathematical models aimed at quantifying the transmission of *C. difficile* in healthcare settings ranging from systems of ordinary differential equations to agent-based models – all developed by undergraduate student researchers! We will discuss how students became involved with their research projects as well as their progress and results. Results can be applied by healthcare professionals by focusing on precautionary measures that reduce patient colonization with *C. difficile*.

Elizabeth DeWitt, Trinity Christian College, *Mathematics and Creative Habits*

Creativity is an in-demand skill in the twenty-first century. To prepare students, institutions of higher education are explicitly including creativity in the curriculum. Yet, in contrast to the perception of mathematicians, many do not view mathematics as a field that develops students' creative thinking. We suggest that mathematics faculty engage the scholarship of creativity experts so that we can help our students communicate their mathematical experiences in language prospective employers connect to creativity. We provide an example mapping between creativity habits as described in Don Perini's text *Emerge* and math instructional practices.

Keith Brandt, Rockhurst University, *Andromeda's Distance: An Activity in Exponential and Logarithmic Functions*

I will describe an activity for Precalculus and College Algebra students based on Hubble's seminal work to determine the distance to the Andromeda galaxy. In the activity, students briefly study background material in on the magnitude (brightness) of stars, find an exponential model, solve exponential and logarithmic equations, and trace through Hubble's calculations for Andromeda's distance. Several practice exercises are included in the activity. Necessary student background is limited to basic familiarity with exponential and logarithmic functions, as would be studied in a typical Precalculus or College Algebra course.

Paula R. Stickles, Millikin University, *Videos and Notes and Calculus, Oh My! A Journey Towards Flipping Calculus*

Calculus serves as a gatekeeper for many students. Often, the obstacle is learning vocabulary and notation amidst the new topics while for others it is the amount of material. How can we engage students in the material from day one? The journey to find an answer continues. In this session we will look at the implementation of required video viewing with a pre-class notes assignment and the impact it appears to have had on student performance. We will share students' reaction to the course structure, preliminary results, lessons learned, and next steps.

Angela Antonou & Christina Jamroz, University of St. Francis, *Research Training Seminar Series - Training Undergraduate Students for Math Research*

This presentation will include an overview of the research training seminar series being offered at University of St. Francis. The goal of the series is to provide undergraduate students some of the skills needed to engage in mathematical research. Specifically, this presentation will discuss the logistics of running the series, suggestions/tips for those wanting to run a similar program at their own institutions, and student feedback on their perceptions of mathematical research after having attended some of the series events.

Elaine Gaberik⁺, Eric Damerow*, Savannah Green*, Joseph Pascual*, Lizbeth Tlatenchi*
Roosevelt University, *A Study of The People's Classifications on MicroPlants*

Zooniverse Unfolding MicroPlant Mysteries is a publicly obtained data experiment to determine if the public is able to distinguish the different characteristics of microplants. After a brief visual and descriptive tutorial, users are asked to classify images of microplants as either “male,” female,” “both,” “sterile,” or “not sure” and then asked to draw boxes over the part of the images the user has identified as either male or female. Other sections have users decide if the branching pattern on a subject is either “regular” or “irregular”, or to decide if leaf structure is “denuded” or “not denuded.” The goal of our team is to work with the raw data of public responses in an attempt to answer the question, can regular, non-expert users overall identify distinguishing traits of the subjects? Which classifications are the public unable to make reliably similar to that of an expert biologist? Using Python scripts, Jupyter Notebook, and Microsoft Excel to clean and analyze the large sets of user data from Zooniverse, our team has put together different tools to answer those questions and visualize the data.

Brendan Miller⁺, Northern Illinois University, *On Low-Rank Convex-Convex Quadratic Fractional Programming*

We present an efficient algorithm for solving fractional programming problems whose objective functions are the ratio of a low-rank quadratic to a positive definite quadratic with convex constraints. The proposed algorithm for these convex-convex problems is based on the Shen-Yu Quadratic Transform which finds stationary points of concave-convex sum-of-ratios problems. We further use elements of the algorithm proposed by Shen and Yu and the classic Dinkelbach approach to ensure convergence. We show that our algorithm performs better than previous algorithms for low-rank problems.

Ronald White⁺, Southern Illinois University, *A Non-Associative Approach to Algebra*

In general, algebra courses focus on building an understanding of algebra from the classical example of the structure of the integers and how they act as a group. We consider what it might be like to build the structure of groups from scratch. Starting with magmas (a set that is simply closed under a binary operation) and build up structure from there. This introduces us to settings (like non-associativity) that is observed in algebraic structures such as Lie algebras. Additionally, since these structures are not as well studied as groups it should to find REU opportunities studying these structures.

Olivia Adamic*, Lewis University, *Mathematically Modeling the Immune Response of Celiac Disease*

Celiac disease is a hereditary autoimmune disease that affects approximately one in 133 Americans. It is caused by a reaction to the protein gluten found in wheat, rye, and barley. After ingesting gluten, a patient with celiac disease may experience a range of unpleasant symptoms while small intestinal villi, essential to nutrient absorption, are destroyed in an immune-mediated process. The only known treatment for this disease is a lifelong gluten-free diet and there is currently no drug treatment. This work uses a system of ordinary differential equations to track changes in small intestinal cell densities. The model can be used to investigate and analyze the immune response by focusing on understanding the dynamics of the small intestine in situations mirroring healthy function and celiac disease. By doing so, we can investigate potential therapies to mitigate the negative effects of celiac disease.

Austin Kind*, Lewis University, *An Agent-Based Model of COVID-19 Transmission at Lewis University*

As new variants emerge, the virus causing COVID-19 continues to spread in the United States and across the world. It is essential to find ways to mitigate its spread, and mathematical models can provide insight into the complex transmission dynamics of COVID-19. In this work, we formulate an agent-based model of the spread of COVID-19 at Lewis University in NetLogo that tracks students' interactions and contamination levels across campus. Using our model, we are able to compare the efficacy of control interventions, such as vaccination and quarantine, in reducing COVID-19 transmission at Lewis.

Keith Gallegos*, Olivet Nazarene University, *The Italian Domatic Number of Trees*

An Italian dominating function of a graph G is a function $f: V(G) \rightarrow \{0, 1, 2\}$ such that for each vertex $v \in V(G)$ either $f(v) \neq 0$, or the sum of weights over all neighbors of v is at least 2. If a family $F = \{f_1, f_2, \dots, f_t\}$ of Italian dominating functions satisfies the condition that the sum weights of a vertex v over all functions in the family is at most 2 for each vertex v , then this is called an Italian dominating family. The Italian domatic number of a graph G , denoted $d_I(G)$, is the maximum cardinality of any such family of distinct Italian dominating functions. This paper considers graphs with small Italian domatic number. In particular, it is shown that a graph must have leaves if it has an Italian domatic number of three, but that determining whether or not a graph has Italian domatic number of three is NP-complete. In addition, trees with an Italian domatic number of three are classified, completely determining the Italian domatic number of trees.

Maximus Lewis* and Laila Mahrat*, Lewis University, *Utilizing an Agent-Based Model to Explore the Transmission of Clostridioides difficile in Healthcare Settings and Evaluate Control Strategies*

Clostridioides difficile (*C. difficile*) is one of the most commonly acquired healthcare-associated infections in United States hospitals. Colonized patients, both symptomatic and asymptomatic, shed *C. difficile* endospores that can survive for long periods of time on surfaces. Transmission pathways include contact with both endospores on fomites, objects likely to carry infection, and

endospore-carrying individuals. In this work, we developed an agent-based model of nosocomial *C. difficile* transmission that focuses on the contribution of environmental pathways in order to examine the role that surfaces with varying touch frequencies contribute to patient colonization. The model incorporates patients, healthcare workers, doctors, room contamination levels separated into high- and low-touch fomites, healthcare worker contamination and decontamination processes, endospore transfer from touch surfaces and person-to-person interactions, as well as patient room and ward cleaning procedures. With this model, we analyzed mitigation strategies individually and in combination for slowing the spread of *C. difficile* transmission, such as increased disinfection rates and cleaning effectiveness of fomites, decreased administration of antibiotics, and increased healthcare worker and doctor hand hygiene compliance levels, including effects such as the use of an alcohol-based handrub versus the use of soap and water. As a result, we determined that transmission was most sensitive to the increased cleaning of low-touch fomites, and without the implementation of more frequent cleaning, these surfaces had consistently higher contamination levels in comparison to those of the high-touch fomites.

Jasmine Gillis*, Roosevelt University, *What Makes a Good Mathematical Proof*

The purpose of the research was to create a method of categorizing mathematical proofs; specifically, what makes a proof that is good in an educational context and what makes a proof good in an applied or theoretical context. During the project 5 proofs of the irrationality of the square root of two and two proofs of the transcendental nature of e were examined using a checklist that highlighted which kind of proof was used (contradiction, construction, induction etc.) what justifications are given, what prior knowledge is needed, complexity level, where the proof is available, and how accessible the language among other observations. It was confirmed that mathematical proofs are heavily impacted by the intended audience and author's background specialty. It follows that reasoning and justifications used within mathematical proofs are similar within educational contexts and professional contexts.

Jakob London*, Lewis University, *Statistical Analysis of High-Pressure Moments in Tennis and Counter-Strike: Global Offensive*

High-pressure sports moments like shootouts, free throws, and serving can greatly impact the outcome of a game. Past research has looked at psychological, physiological, and behavioral factors in an attempt to understand the phenomena of "choking under pressure." In the summer of 2022, statistical analysis research was performed to further understand these high-pressure moments specifically in tennis and the esport Counter-Strike: Global Offensive. The goals of this research were to further understand these high-pressure moments specifically in tennis and combat/fighting esports games. We explored questions like how do different professional players perform under pressure? Are top players less likely to choke? Are there any similar trends between tennis players and competitive esport players and how they respond to high-pressure situations within their sport? In search of assurance that this research had been performed without fault, a framework for formally verifying both this statistical analysis and others like it was also proposed. The formal verification framework introduced consists of employing a translator to convert statistical analysis code written in Python into the B specification language.

Once the Python code has been translated into the B language, then the ProB model checker can perform formal verification on the constructed B machine.

Viet Bui*, Augustana College, *Computing the period of the Simple Pendulum using Elliptic Integrals*

The simple pendulum is a classic example of harmonic oscillation that's used in fields such as physics, engineering, and mathematics. In this presentation, we discuss the derivation of the differential equation for the equation of motion ($F = ma$). We then find the period of the simple pendulum using an elliptic integral. We also use numerical methods to build a simulation model to visualize the Pendulum's movements in Python.

Shiqi Cheng" and Jibrán Haque*, Benedictine University, *The Isomorphy Classes of the Special Orthogonal Group of Low Dimension in Characteristic 2*

In 1926, Cartan introduced the concept of real symmetric spaces as a class of homogeneous Riemannian manifolds, where a symmetric space is defined as the homogeneous space G/H with G a reductive group and $H = G^\theta = \{g \in G \mid \theta(g) = g\}$, the fixed-point group of an n -automorphism. Of special interest are automorphism of order 2, also called involutions. In this project we focused on classifying the involutions for the $SO(3, \mathbf{F}_{2^k})$ and $SO(4, \mathbf{F}_{2^k})$. Our results should provide a base case for the general theory for n -dimensional matrices.

Career Panel--Saturday

Career Panel for Undergraduate Students: Preparation for Careers in Mathematical Sciences

Ayesha Quadri is a Treasury Associate at Revantage, works on cash forecasting, modeling and maintaining quarterly distribution analyses for their companies. Graduated in 2017 with a BA in Mathematics and Actuarial Science from Benedictine University.

Bisma Mirza, is a Product Owner at Allstate Insurance where she works with two teams of Tableau Data Visualization Developers to provide data driven reporting and insights to company leaders regarding claims. Graduated in 2015 with a BS in Mathematics from Benedictine University.

Emma Schoonover teaches High School Math at Ottawa Township High School for the past three years and Head Girls Track Coach for the last two years. Currently also a graduate student in STEM Education, American College of Education. Graduated with a BA in Math with a minor in Secondary Education from Benedictine University.

Section NExT Round Table Discussion—Saturday

What issues do new college math teachers face in Illinois today? Between advising students from a variety of backgrounds, considering different forms of assessment, fitting in time for research,

and taking on service tasks, college math teachers face many challenges. In our round-table discussion, fellows of the ISMAA Section NExT program will be able to ask questions and get feedback early in their career. If you are interested in adding to the discussion, or asking the group questions of your own, please join us! We are fortunate to be able to constantly learn from each other. Moderated by **Emily Olson**, ISMAA Section NExT Coordinator.

Contributed Talks—Saturday

Keven Hansen, Lake Forest College, *Illinois Developmental Education Reform--What Every Math Department Needs to Know*

In recent years, the math curriculum in many states has been shaped by legislative action. This is particularly true regarding placement and developmental education. Illinois now has laws impacting mathematics departments at public community colleges and universities in each of these areas; this session will provide information about the laws, their impact, and how various schools are addressing these challenges. An opportunity to share concerns, questions, and successes will be provided.

Matthew Timm, Bradley University, *Coset Labeled Graphs*

We introduce the notion of a coset labeled graph. These graphs provide a nice introduction to group actions on graphs. We show how to use them to build topologically interesting spaces.

Daniel Hess & Sarah Ziesler, University of Chicago, *A Learning Community for Instructional Faculty*

To grow as educators, we need regular time put aside to reflect on our teaching, to learn about new approaches that others have tried (whether successful or not), and to explore the Scholarship of Teaching and Learning. In response to this need, the Pedagogy Seminar was established in the mathematics department at the University of Chicago, in Autumn 2020. In this presentation, we will discuss the development of the Pedagogy Seminar, its challenges and successes, and our hope for where it will go next.

M. Vali Siadat, Richard J. Daley College, *Innovations in Teaching and Learning of Mathematics*

We present the theory and practice of a research-based model of teaching and learning of mathematics at the college, coined the Keystone model. This paper discusses a multi-year research in mathematics education of undergraduate college students in the U.S. and focuses in the first two years in the undergraduate mathematics curriculum. The research reports on two areas of student learning outcomes: performance in mathematics and improvements in students' concentration skills. Under the application of the program, students not only achieved significant outcomes in mathematics, as evidenced in formative and summative assessments, but also improved in their concentration skills as demonstrated in the gains in national reading comprehension tests. The gains achieved for the participants in the study were all statistically significant, compared to the control groups.

Yina Wang, Illinois Math and Science Academy, *The Effects of Vaccine Misinformation on Disease Spread*

In the spring of 2021, vaccines from Pfizer, Moderna, and Johnson and Johnson became available to the general public to help protect against COVID-19. Since then, a total of 69% of the eligible population in the U.S. has been fully vaccinated against the disease, but this still leaves about 70,000,000 completely unvaccinated. One reason for this has been the spread of misinformation regarding the vaccines, ranging from the seemingly plausible (vaccine affects fertility, COVID isn't that bad of a disease) to the absurd (vaccine will implant 5G microchip). Although all of this misinformation has been debunked in some form or another, it persists and can form the basis for people refusing to take the vaccine.

In this talk, we will build a mathematical model for the spread of a disease through a population where a vaccine is available. We will then introduce an active misinformation spread to try and answer the question: What are the short- term and long-term effects of vaccine misinformation on the spread of a disease where a vaccine is available? Is it simply a nuisance, or does it have a profound effect on the disease spread? We address this via numerical simulations as well as more general analysis using theoretical techniques (Jacobian and next-generation matrices).

Kiah Wah Ong, Illinois Institute of Technology, *Hysteresis in an insect outbreak model*

Dynamical systems often depend on parameters and the behavior of a dynamical system can change drastically when the parameter crosses a critical threshold. We say that hysteresis occurs when a simple reversal of the parameter after crossing the threshold does not get the system back to its initial state. In this talk, we will use a simple insect outbreak model to demonstrate this phenomenon.

Sofia Briggs*, Augustana College, *Counting subgroups of finite Abelian p -groups*

We consider groups of the form $Z_p \times Z_{p^2} \times \dots \times Z_{p^n}$ and count the number of subgroups of order p^k using recurrence relations. We compare our results to computations done with SAGE and the previous research of Marius Tarnauceanu. Our method counts both cyclic and noncyclic subgroups, which builds on Tarnauceanu's previous work for counting cyclic subgroups.

Laila Mahrat* and Matthew Senese*, Lewis University, *Stochastic Simulations of Nosocomial *Clostridioides difficile* Infections and Transmission*

Clostridioides difficile (*C. difficile*) infection has been a prominent issue in healthcare settings for over a decade with healthcare workers (HCWs) and fomites, surfaces likely to carry infections, being recognized as the main vectors of transmission. We are developing an agent-based model (ABM) to simulate interactions between patients, HCWs, and fomites in a six ward hospital. Due to an ABM's reliance on probabilities, our model generates stochastic interactions that simulate real-world scenarios. Additionally, we can monitor characteristics of our agents, such as patient disease status or contamination level of HCWs to determine the current threat level of a disease outbreak. We will explore the factors that most lend towards disease transmission and test new strategies to reduce the spread of *C. difficile* in healthcare settings.

Steve Cohen, Roosevelt University, *Essential Student Learning Outcomes for the College Geometry Course for Secondary Teachers*

Abstract. What should be in the College Geometry class taken to prepare preservice high school math teachers to teach geometry? To work towards agreement, an online community of faculty who teach or do research on college geometry courses taken by secondary teachers (GeT courses) was formed in Summer 2018 by Pat Herbst and Amanda Brown from the University of Michigan GRIP lab (<https://www.gripumich.org/>). A subgroup of GeT instructors in the project worked on articulating a set of essential student learning outcomes (SLOs) where essential means the identification of content knowledge that all prospective secondary geometry teachers should have the opportunity to learn. This presentation reports on the first public draft of the set of Student Learning Outcomes, available at <https://getapencil.org/student-learning-objectives/> and ongoing efforts to disseminate the SLOs in order to gather feedback and wider perspectives on the SLOs.

Rachel Rupnow, Northern Illinois University, *Same Up to What? Reflections on Ways Sameness Permeates Mathematics*

Sameness appears throughout mathematics via particular concepts like equality, isomorphism, and congruence, but also underlies approaches to core questions in mathematics related to classification and uniqueness. In this talk, I will explore descriptions of sameness given by mathematicians and students as well as key features to consider when determining whether objects are the same in context. Using these descriptions, I will ask audience members to think about how they would respond to hypothetical student questions from linear algebra, discrete math, and abstract algebra to more deeply consider how we, as mathematicians, communicate what to pay attention to and can direct students' attention to important features.

Brian Freeman* & Bushra Ibrahim*, Millikin University, *Analyzing the Game of Cycles using Python*

The Game of Cycles is a game that is played by two players on a simple connected planar graph, called a board. Each player takes turns directing edges on a board with an arrow without allowing for a source or sink to occur. The game continues until a player creates a directed cycle cell or makes the last possible move. In this talk, we will use Python to simulate the game and examine optimal winning strategies on boards involving tree graphs.

Ricardo Esteban Salazar Ordoñez*, Lake Forest College, *Approximating The best building block set*

We study the well-known children toy "Building blocks", a set of cubes with colored letters or shapes on their sides, whose benefits for children are the development of motor-skills and dexterity, as well as problem solving, among others perks. The end goal is to create a 30-block set, that permutes 6 colors among the letters of the English alphabet, to maximize the number of words that can be spelled, where the letters in the words have either a different color each (rainbow words) or, that they all share the same color (mono-color words). Since the search space for this problem is intractable, we are approaching this research with simulated annealing,

which despite not allowing us to state for sure that the permutation found is the optimal solution, it will be one that is better than many other solutions.