Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
March 2020

These problems were prepared for the 2021 ISMAA meeting which was held virtually, with the result that there was no 2021 Contest. We encourage **anyone or any group** of students and/or faculty to consider these problems and, if solved, write up a solution and send it to us. Think of these much like problems from the *Monthly*. In a few weeks we will publish solutions along with the names of all those who submitted correct solutions. This will be posted on the ISMAA web site. Please send solutions – with identifying names and institutions – electronically (preferably) to `pgandrews@eiu.edu` or by mail to Peter Andrews, Department of Mathematics and Computer Science, Eastern Illinois University, 600 Lincoln Ave., Charleston, IL, 61920 so that we receive then by Monday, April 11, 2022. Happy problem solving!!

1. **Moving Cubes** Two thousand identical small metal cubes are placed on a horizontal table, with 1000 on the left side and 1000 on the right side. Cubes are moved from side to side according to the following scheme: after one minute, 100 cubes have been moved from the left to the right; after another minute, 101 cubes have been moved from the right to the left; after another minute 100 cubes have moved from the left to the right; after the next minute 101 cubes have moved from right to left. In other words, alternating minute by minute, 100 balls from left to right then 101 balls from right to left, starting with the left to right move. The process will stop if there are not enough balls on one side or the other to make the required move.

   (a) Demonstrate that there will be a time (after the initial configuration) when there will be exactly 1000 balls on each side of the table and determine the first such time.

   (b) Will this happen again? If so, determine the next time that happens. If not, prove why not.

2. **Don't Solve Quadratic** If x is a positive number so that $x - \dfrac{1}{x} = \sqrt{51}$, without finding x itself, find the exact values of $A = x^2 + \dfrac{1}{x^2}$ and $B = x^2 - \dfrac{1}{x^2}$.

3. **Nested Boxes** There is an empty box. Put in the empty box, side by side with each other, k empty boxes. (Now the first box is no longer empty.) At the next step, and each step thereafter, for each empty box, **either** place k empty boxes side by side inside it, making it no longer empty, **or** leave the box empty. At the end m boxes were non-empty and N boxes were empty.

   (a) When there are $m = 100$ non-empty boxes, how many empty boxes, N, are there if $k = 4$?

   (b) When there are $N = 100$ empty boxes, how many non-empty boxes, m, are there if $k = 4$?

   Now change the number of empty boxes placed inside each box from 4 to 5.

   (c) When there are $m = 75$ non-empty boxes, how many empty boxes, N, are there if $k = 5$?

   (d) Is it possible that there could be $N = 100$ empty boxes if $k = 5$? If so, how many non-empty boxes, m, are there at that point? If not, why not?

4. **Ten Coins and the Balance** Ten coins lie on a table. The weight of each coin is a whole number of grams in the range from 1 to 100 grams inclusive. We want to put some of these coins on the left pan of the balance and some on the right pan so that the balance is in equilibrium (some coins may remain on the table). Is it always possible to do? If it's always possible, give a proof. If not always, give an example of the 10 weights that can't be split this way.

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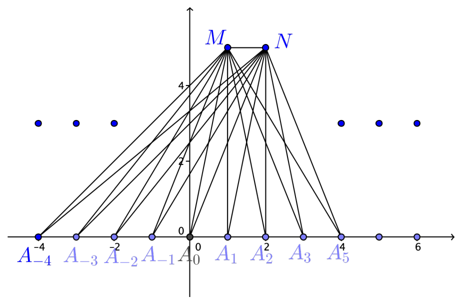
5. **The Sum of the Angles** All the integer points $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ on the x-axis are labeled, respectively, as $\ldots, A_{-3}, A_{-2}, A_{-1}, A_0, A_1, A_2, A_3, \ldots$. Segment MN of length 1 is situated at some place on the horizontal line $y = 5$. Consider all the angles $\angle MA_iN$, where the integer index i runs from $-\infty$ to $+\infty$. What can the sum $S = \displaystyle\sum_{i=-\infty}^{+\infty} \angle MA_iN$ of all these angles be equal to? Does this sum depend on the position of segment MN on the line $y = 5$?



6. **First Digits of 2^n and 5^n** Integers $2^5 = 32$ and $5^5 = 3125$ have the same first (the leftmost) digit 3. It can be proven (but not that easily!) that for some integer exponent n, the integers 2^n and 5^n have the same 10 first (leftmost) digits and there is only one possible such 10-tuple. Determine these 10 digits.