The Enjoyment of Elementary Geometry

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A Well-known Exercise

(a) As appeared in a trigonometry textbook (Larson, Hostetler, & Edwards):

A photographer is taking a picture of a 4-foot-tall painting hung in an art gallery. The camera lens is 1 foot below the lower edge of the painting. The angle β subtended by the camera lens x feet from the painting is

$$\beta = \arctan\frac{4x}{x^2+5},\ x > 0.$$

Use a graphing utility to graph β as a function of x. Move the cursor along the graph to approximate the distance from the picture when β is maximum.

(b) As appeared in a calculus textbook (Larson, Hostetler, & Edwards):

A photographer is taking a picture of a 4-foot painting hung in an art gallery. The camera lens is 1 foot below the lower edge of the painting. How far should the camera be from the painting to maximize the angle subtended by the camera lens?

Solution. Let β and x be as above. Then $\beta = \text{arccot}\frac{x}{5} - \text{arccot}x$. So

$$\frac{d\beta}{dx} = \frac{-1/5}{1+\frac{x^2}{25}} - \frac{-1}{1+x^2} = \frac{-5}{25+x^2} + \frac{1}{1+x^2} = \frac{4(5-x^2)}{(25+x^2)(1+x^2)}.$$

Setting $\frac{d\beta}{dx} = 0$ and applying the First Derivative Test, you can conclude that $x = \sqrt{5}$ yields a maximum value of β.

The Origin of the Problem

In 1471 Regiomontanus posed the following problem in a letter to Christian Roder, a Professor at Erfurt:

At what point on the ground does a perpendicularly suspended rod appear largest (i.e., subtends the greatest visual angle)?

Regiomontanus

- Born in Unfinden, near the town of Königsberg in Franconia, in 1436. Died in 1476 in Rome.

He was a man known by many names:

Johann Müller

Johannes Germanus (because he was a German)

Johannes Francus (because Franconia was known as Eastern France)

Johann von Kunsperk (after the town of Königsberg)

Regio Monte (Latin translation of Königsberg, "the royal mountain")

Joannes de Monte Regio

Regiomontanus

He is best remembered for the book: *De triangulis omnimodis* (*On Triangles of Every Kind*), written in 1464, printed in 1533. We quote a wonderful paragraph from the introduction of this book: "You, who wish to study great and wonderous things, who wonder about the movement of the stars, must read these theorem about triangles. ... For no one can bypass the science of triangles and reach a satisfying knowledge of the stars. ... A new student should neither be frightened nor despair. ... And where a theorem may present some problem, he may always look down to the numerical examples for help."

A Geometric Solution

Suppose that the upper and lower edges A and B of the painting are a and b feet above the horizontal line l at the level of the camera lens. Draw the circle through A, B and tangent to l, then it is easy to see that β is maximum at the tangency point. By the power of a point, this optimal distance is $x = \sqrt{ab}$.

VIP Seats in a Multiplex Stadium-seating Movie Theater

Once in my class, I phrased the question as finding the best parking spot in a drive-in theater to maximize the (vertical) viewing angle. It is then only natural to ask:

Where are the best seats in a multiplex stadium-seating theater to maximize the viewing angle of your favorite Hollywood stars, car chases, explosions and other special effects?

The geometric solution is still valid, with l changed to the tilted line joining the audience's eye-levels.

A Ruler-compass Construction

Note that $ab = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2$. So $x = \sqrt{ab} = \sqrt{(\frac{a+b}{2})^2 - (\frac{a-b}{2})^2}$, which can be constructed by Pythagorean Theorem.

Another Ruler-compass Construction from the CRUX

Problem 2822 [2003, 114]. *Proposed by Peter Y. Woo, Biola University, CA*

Suppose that Π is a parallelogram with sides of lengths $2a$ and $2b$ and with acute interior angle α, and that F and F' are the foci of the ellipse Λ that is tangent to the four sides of Π at their mid-points.

(a) Find the major and minor axes of Λ in terms of a, b, and α.

(b) Find a straight-edge and compass construction for F and F'.

Solution.

(a) Let E be the ellipse $\frac{u^2}{a^2} + \frac{v^2}{(b\sin\alpha)^2} = 1$ and R the rectangle tangent to E at its u, v-intercepts.

Under the affine change of coordinates

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & -\cot\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

with $\cot 2\theta = \frac{a^2+b^2\cos 2\alpha}{b^2\sin 2\alpha}$, R is transformed to Π and E to Λ with equation $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, where $A = \sqrt{\frac{a^2+b^2+C^2}{2}}$, $B = \sqrt{\frac{a^2+b^2-C^2}{2}}$, and $C = \sqrt[4]{a^4 + 2a^2b^2\cos 2\alpha + b^4}$.

(b) By the law of cosines, $p = \sqrt{a^2 + b^2 - 2ab\cos(\frac{\pi}{2} + \alpha)}$ and $q = \sqrt{a^2 + b^2 - 2ab\cos(\frac{\pi}{2} - \alpha)}$ can be constructed. Now, we notice that $C = \sqrt{pq} = \sqrt{(\frac{p+q}{2})^2 - (\frac{p-q}{2})^2}$. So we can construct C, thus $A = \sqrt{\frac{a^2+b^2+C^2}{2}}$ and $B = \sqrt{\frac{a^2+b^2-C^2}{2}}$ as well.

Finally, let $r > s$ be the distances from F, F' to the midpoint M of a side of Π of length $2a$. Then $r + s = 2A$ and $r^2 + s^2 = 2(b^2 + C^2)$. So $r - s = \sqrt{2(r^2 + s^2) - (r + s)^2} = 2\sqrt{b^2 + C^2 - A^2}$. Hence $r = A + \sqrt{b^2 + C^2 - A^2}$ and $s = A - \sqrt{b^2 + C^2 - A^2}$ can be constructed, completing the solution.

A MONTHLY Problem

The tangent-circle solution inspired me to find a geometric solution to another problem.

Problem 10904 [2001, 871 & 2002, 860-861]. *Proposed by Ovidio Furdui, Western Michigan University, MI.*

(a) Let $ABCD$ be a convex quadrilateral. Prove that if there is a point in the interior of $ABCD$ such that $\angle PAB = \angle PBC = \angle PCD = \angle PDA = 45°$, then $ABCD$ is a square.

(b) Generalize from squares to n-gons.

Solution to (a). Let E and G be the feet of the perpendiculars from P to AB and CD, respectively. Locate F and H so that $AEPF$ and $CGPH$ are squares. Let Γ_1 be the circle with center F and radius FA, and Γ_2 the circle with center H and radius HC. Then D must be on Γ_1 and B must be on Γ_2, in order to satisfy $\angle PDA = \angle PBC = 45°$. Therefore, $FI \leq FA$ and $HJ \leq HC$, where I and J are the feet of the perpendiculars from F and H to CG and AE respectively. If $CG \parallel EA$, then $FI \leq FA = HJ \leq HC = FI$, which forces all the equalities to hold. Thus $I = D$ and $J = B$, and $ABCD$ is a square. Otherwise, without loss of generality, we may extend EA beyond A to intersect CG beyond G at K. Then $FA < HJ \leq HC$, and therefore $FI + HJ = HC + FA + (HC - FA)\sin\angle CKE > HC + FA$. This contradiction completes the proof.

Erdös-Mordell-Barrow Inequality

The case of $n = 3$ of the above MONTHLY problem is also true. In fact, it is an immediate consequence of the famous Erdös-Mordell-Barrow Inequality:

Let P be a point in the interior of a triangle ABC. Let r, s, and t be the distances from P to the vertices, and let x, y, and z be the distances from P to the sides. Then

$$r + s + t \geq 2(x + y + z),$$

with equality if and only if ABC is equilateral.

Proof. Let E and F be the feet of the perpendiculars from P to CA and AB. By the law of cosines, $EF^2 = y^2 + z^2 - 2yz\cos\angle EPF = y^2 + z^2 + 2yz\cos A = y^2 + z^2 - 2yz\cos(B + C) = (y\sin C + z\sin B)^2 + (y\cos C - z\cos B)^2 \geq (y\sin C + z\sin B)^2$. Hence $r = \frac{AE}{\sin\angle APE} = \frac{AE}{\sin\angle AFE} = \frac{EF}{\sin A} \geq y\frac{\sin C}{\sin A} + z\frac{\sin B}{\sin A}$. By symmetry, $s \geq z\frac{\sin A}{\sin B} + x\frac{\sin C}{\sin B}$ and $t \geq x\frac{\sin B}{\sin C} + y\frac{\sin A}{\sin C}$. By the AM-GM inequality $\frac{\sin C}{\sin A} + \frac{\sin A}{\sin C} \geq 2$, etc., completing the proof.

Brocard Points

This special case of $n = 3$ leads us further to the Brocard points. They are named after Henri Brocard, a French army officer, who described them in 1875. However, they had been studied earlier by Jacobi, and also by Crelle, in 1816. Crelle was led to exclaim:

"It is indeed wonderful that so simple a figure as the triangle is so inexhaustible in properties. How many as yet unknown properties of other figures may there not be?"

For any triangle there is a unique angle ω, the *Brocard angle*, and points Ω and Ω', the *Brocard points*, such that $\omega = \angle\Omega AB = \angle\Omega BC = \angle\Omega CA = \angle\Omega' AC = \angle\Omega' BA = \angle\Omega' CB.$

A Dog Chase in a Triangle

To find the Brocard points, put three dogs at the vertices of the triangle and let them chase one after the other at the same speed. Depending on the directions of chasing, clockwise or counter-clockwise, they will end up at the corresponding Brocard points.

The Most Important Fact about the Brocard Angle ω

$$\cot\omega = \cot A + \cot B + \cot C$$

An Angle of Many Inequalities

For example,

(a) $\omega \leq 30°$ (equality iff ABC is equilateral);

(b) $\omega^3 \leq (A-\omega)(B-\omega)(C-\omega)$;

(c) $2\omega \leq \sqrt[3]{ABC}$ (the Yff inequality);

(d) $2\omega \leq \frac{3}{\frac{1}{A}+\frac{1}{B}+\frac{1}{C}}$.

An IMO Problem

IMO Problem 1991/5. Let ABC be a triangle and P an interior point in ABC. Show that at least one angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

Solution. P must be in one of the three smaller triangles ΩAB, ΩBC, or ΩCA. Since $\omega \leq 30°$, the proof is complete.

A Recent MONTHLY Problem

A recent MONTHLY problem offers a new inequality of the Brocard angle.

Problem 11017 [2003, 439]. *Proposed by C. R. Pranesachar, Indian Institute of Science, India.*

Let P and Q be the Brocard points of a non-equilateral triangle T, and let ω be the Brocard angle of T.

(a) Prove that the line through P and Q passes through a vertex of T if and only if the sides of the triangle are in geometric progression.

(b) Prove that $\omega <$ min$[\frac{\pi}{6}, \frac{B+C}{3}]$, where B and C are the smallest angles of T.

Solution to (b). It suffices to show that $\omega < \frac{B+C}{3}$ if $B+C < \frac{\pi}{2}$. Since cot$\omega =$ cot$A +$ cot$B +$ cot$C$ and cot$x$ is a decreasing function of $x \in (0,\pi)$, we only need to show that cot$\frac{B+C}{3} <$ cot$(\pi - B - C) +$ cot$B +$ cot$C$, or equivalently, cot$\frac{B+C}{3} <$ cot$B +$ cot$C -$ cot$(B+C)$. Using the Euler product sin$x = x\prod_{n=1}^{\infty}[1-(\frac{x}{n\pi})^2]$, we obtain cot$x = \frac{d}{dx}$(lnsin$x) = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{1}{n\pi}(\frac{1}{1+\frac{x}{n\pi}} - \frac{1}{1-\frac{x}{n\pi}}) = \frac{1}{x} + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n\pi}[(-\frac{x}{n\pi})^k - (\frac{x}{n\pi})^k] = \frac{1}{x} - \sum_{k=0}^{\infty}s_k x^{2k+1}$, where $s_k = \sum_{n=1}^{\infty}\frac{2}{(n\pi)^{2k+2}}$. Hence, cot$B +$ cot$C -$ cot$(B+C) -$ cot$\frac{B+C}{3} = \frac{1}{B} + \frac{1}{C} - \frac{4}{B+C} + D$, where $D = \sum_{k=0}^{\infty}s_k[(1+\frac{1}{3^{2k+1}})(B+C)^{2k+1} - B^{2k+1} - C^{2k+1}] > 0$. Finally, by the AM-HM inequality, $\frac{B+C}{2} \geq \frac{2}{\frac{1}{B}+\frac{1}{C}}$, thus $\frac{1}{B} + \frac{1}{C} \geq \frac{4}{B+C}$, completing the proof.

Back to the Trigonometric Exercise

Of course, $\frac{1}{B} + \frac{1}{C} \geq \frac{4}{B+C}$ is merely another expression of

$(B+C)^2 - 4BC = (B-C)^2 \geq 0$, with equality iff $B = C$.

Recall that $\beta =$ arctan$\frac{4x}{x^2+5}$. It suffices to maximize $\frac{4x}{x^2+5}$.

Note that $\frac{4x}{x^2+5} = \frac{4}{x+\frac{5}{x}} \leq \frac{1}{x} + \frac{x}{5}$, with equality iff $\frac{1}{x} = \frac{x}{5}$, i.e. $x = \sqrt{5}$.

This kind of non-calculus ideas seemed to be what Regiomontanus used to solved his problem. It is not clear whether he actually provided a solution.