Malfatti-Steiner Problem

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Abstract
Based on Julius Petersen's work we give detailed proofs of his statements on the proof of Steiner's conjecture.

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Malfatti problem was posed by the Italian mathematician Giovanni Francesco Malfatti (1731-1807) in 1803.
The problem has two aspects. One is to maximize the area of three circles within a triangle. Malfatti thought that the answer was three mutually touching circles which are also tangent to the sides of the triangle.This was later shown to be incorrect (Lob and Richmond 1929).
The other is to give a Euclidean construction of these three circles. Malfatti gave only an analytic solution, by calculating the radii of the circles. The key to the solution was found by Jacob Steiner (1796-1863). Steiner did not prove his discovery. It is an amazing and very unlikely discovery. One might think that he perhaps guessed it by a construction. But it is such an intriguing relation that even after one knows it, it is not easy to see it, even with a large and exact construction. Steiner also indicated the way by which his conjecture could be proved. The proof was given according to some sources by Hart in 1856 and others (Julius Petersen) by Schroeter in 1874. No matter who gave the proof first, the long time interval (30 , or 48 years) between the conjecture and the proof is a testimony to the complexity of the problem.
Here I would like to look at the solution of Julius Petersen (1839-1910). This Danish geometer is best known for his small problem book with about 400 problems, many of which are challenging.At the end of his book he outlines in a few lines his "simpler" solution. There is not even a single figure in the French translation I used.
This paper is based on his work. I tried to fill in all the needed proofs to clarify the construction in my mind. Most of them are based on inversion. They may or may not be what Peterson had in mind.
I start with a simple figure and every time an element of the proof is introduced I give a new figure. This way when the figure finally gets very complex the reader hopefully will be familiar with all the elements of the last figure. I also give separate figures for the proofs by inversion.

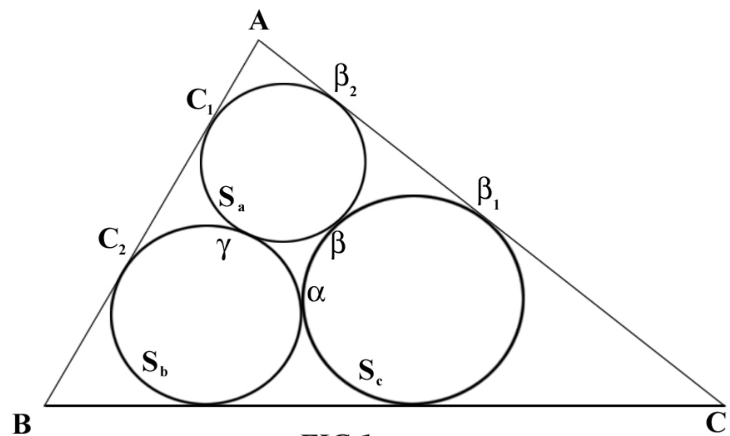

FIG.1

Fig.1 Take the triangle ABC.Consider the problem solved.

Call the circles S_a, $S_{b,}$ and S_c .

Let their mutual contact points be a , ß , and ?.

Let the points where they touch the sides AB and CA be C_1, C_2 and $ß_1$, $ß_2$

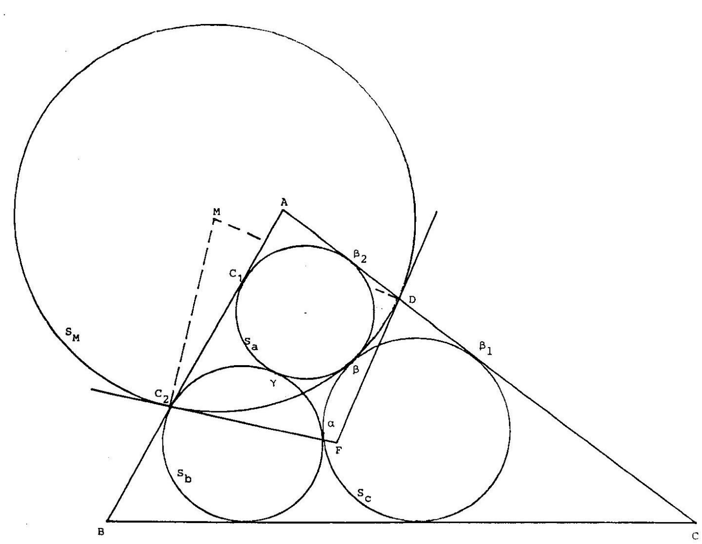

Fig.2

Fig.2 Draw the circle which is tangent to the circles S_a , and S_b at their contact point

ß and goes through the point C_2 .

Call this circle $S_{M,}$ and its center M.

Call the intersection of S_M with the side CA D.

Draw two tangent lines to the circle SM at points C_2 and D.

Call the intersection point of these tangent lines F.

We claim that the circle S_M intersects the circle S_b and the side CA under equal angles.

That is, the angle $\angle AC_2F$ is complementary to the angle $\angle FDA$.

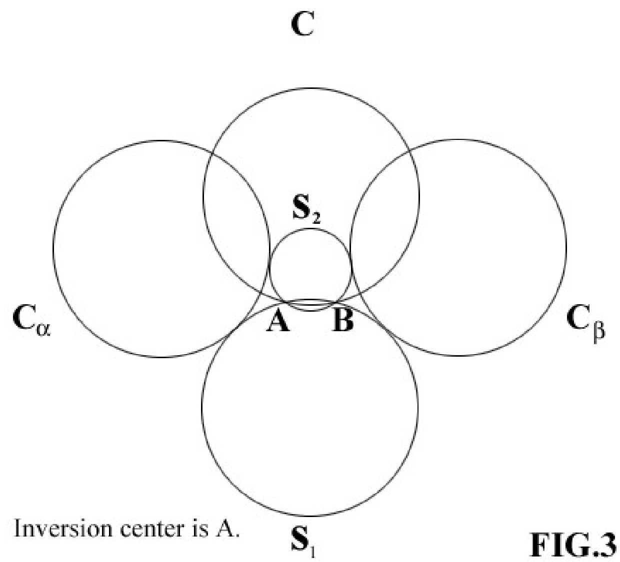

FIG.3

Fig.3 Proof:

Theorem:

Given two circles $S_{1,}$ and S_2 which intersect each other at points A, and B.

Any circle C which goes through A and B intersects all circles like C_a, and $C_ß$ which are tangent to S_1, and S_2 under equal angles. (Complementary angles if the circles are on opposite sides.)

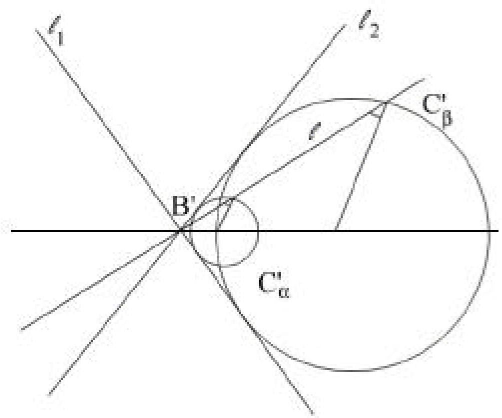

FIG.4

Fig.4 We invert the system with respect to the inversion center A. With this inversion

S_1 becomes a line l_1
S_2 becomes a line l_2
C becomes a line l

The point A, being the inversion center goes to infinity.

The point B becomes B'.

The lines l_1, l_2, and l go through the point B', because the circles S_1, S_2, and C go through the point B.

The circles C_a and $C_{\text{ß}}$ become again circles C'_a, and $C'_{\text{ß}}$ since they don't go through the inversion center A.

The circles C'_a, and $C'_{\text{ß}}$ are tangent to the lines l_1 ,and l_2 because the circles C_a, and $C_{\text{ß}}$ were tangent to the circles S_1, and S_2 before the inversion.

From the inverted drawing it is seen that the line l intersects the circles C'_a, and $C'_{\text{ß}}$ under equal angles .

Since the inversion does not change angles, the uninverted circle C intersects the uninverted circles C_a, and $C_{\text{ß}}$ under equal angles. (If C_a, and $C_{\text{ß}}$ are on opposite sides, the angles are complementary.)

In our case the circles S_1, and S_2 are S_a, and $S_{c.}$ The points A and B coincide at the point ß, since in our case S_a, and S_c are tangent.

The tangent circles C_a, and $C_ß$ are in our case S_b, and the line CA (degenerate circle with its center at infinity).

The circle C is in our case S_M.

Thus we showed that the circle S_M intersects S_b, and CA under complementary angles.

That is $\angle FC_2A + \angle FDA = 180^o$

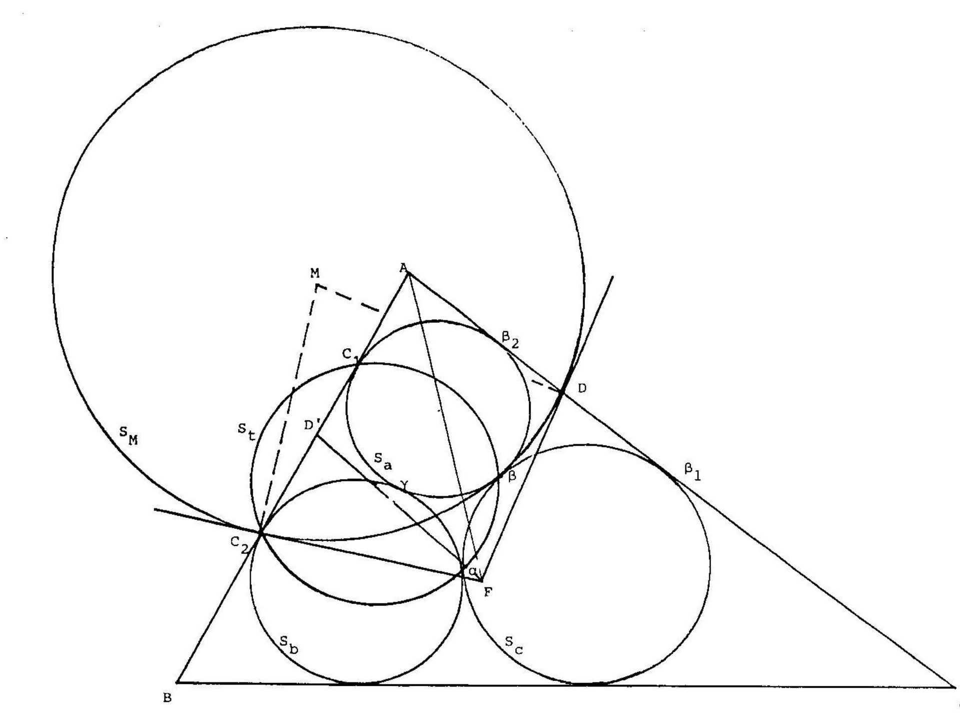

Fig.5

Fig.5 Consider the quadrilaterals MC_2FD and MC_2FD.

We just proved that the opposed angles at C_2 and D in the quadrilateral AC_2FD add up to 180^o.

Now in the quadrilateral MC_2FD the tangent C_2F is $\perp$ to the radius MC_2.

Thus the angle $\angle MC_2F = 90^o$.

Similarly FD $\perp$ MD. Thus the angle $\angle FDM = 90^o$.

Hence the sum of these two opposite angles is 180^o.

Also, the two quadrilaterals have the angle $\angle C_2FD$ common.

Thus $\angle C_2MD = \angle C_2AD$.

Now connect A to F. We will prove that AF is the bisector of the angle $\angle C_2MD = \angle A$.

Let D' be the point on AB with $FC_2 = FD'$.
The triangles AD'F = ADF. Because:

1) AF = AF is common to both triangles.
2) FD = FD' .Because $FD=FC_2$ are tangents to the circle S_M and we took $FD' = FC_2$.
3) $\angle AD'F = \angle ADF$. Because the triangle C_2FD' being isosceles $\angle ADF$ is complementary to $\angle AC_2F$. But we also showed that $\angle FDA$ is complementary to $\angle AC_2F$.

Hence AD'F = ADF

Fig.5 We now claim that the points a, ß, C_1, and C_2 lie on a circle . We shall call this circle S_t and prove the claim below.

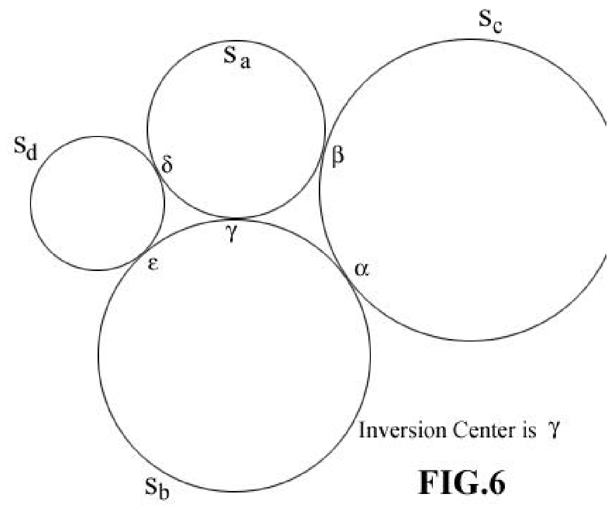

FIG.6

Fig.6 Theorem:

Given two mutually tangent circles S_a, and S_b which touch at ?. Consider the circles S_c, and S_d which touch S_a, and S_b ats the points a, ß, d, and e. The points a, ß, d, and e lie on a circle.

The proof is again by inversion. Take ? as the center of inversion. S_a, and S_b become under inversion two lines l_a, and l_b. the point ? goes to infinity. Since S_a, and S_b do not have a second intersection point (they are tangent) they become parallel lines under inversion.

S_c, and S_d are tangent to both S_a, and S_b. therefore their images must be tangent to the inverted images of S_a, and S_b.

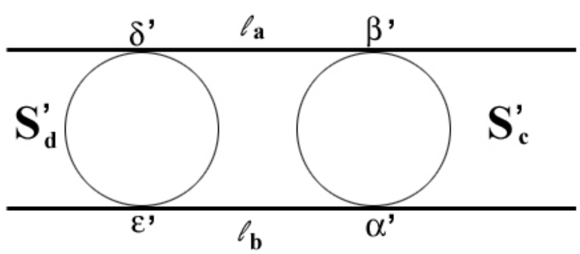

FIG.7

Fig.7 Obviously the points a', ß', d', and e' lie on a circle. Because they lie at the corners of a rectangle. Hence the inverted image of this circle goes through the points a, ß, d, and e.

In our case S_d is degenerate (it is the line C_1C_2 tangent to the circles Sa, and S_b). Thus we established that the points a, ß, C_1, and C_2 lie on a circle , we called S_t.

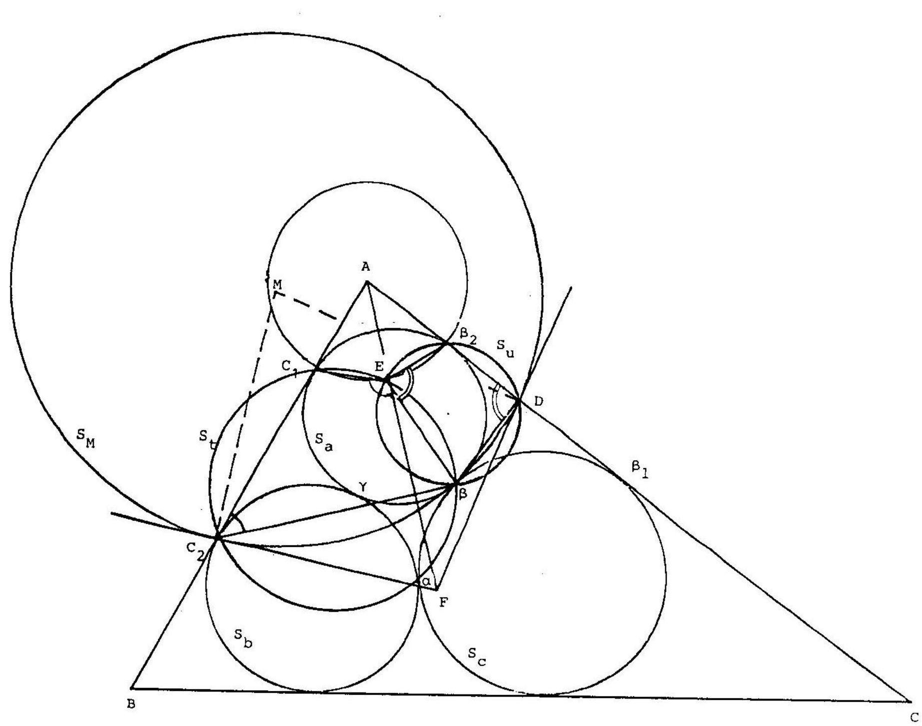

Fig.8

Fig.8 Draw the circle through ß, D, and $ß_2$. Call this circle S_u.
S_u intersects S_t at E.
Connect C_1 to E, also $ß_2$ to E.
Connect C_2 to ß, also E to ß.
We will prove that E lies on the circle with the center A, which goes through C_1, and $ß_2$.

From the circle S_t $\angle C_1Eß = 180^o - \angle C_1C_2ß$

From the circle S_u $\angle ß_2Eß = 180^o - \angle ß_2Dß$

Adding $\angle C_1Eß + \angle ß_2Eß = 360^o - (\angle C_1C_2ß + \angle ß_2Dß)$

But $\angle C_1Eß + \angle ß_2Eß = 360^o - \angle C_1Eß_2$

Hence $\angle C_1Eß_2 = \angle C_1C_2ß + \angle ß_2Dß$

But $\angle C_1C_2ß + \angle ß_2Dß = 360^o - \angle A - \angle C_2ßD$

$\angle C_2ßD$ is an angle in the circle S_M and $\angle C_2ßD = 180^o - \frac{1}{2}\angle A$

Finally $\angle C_1Eß_2 = 360^o - \angle A - 180^o + \angle \frac{1}{2}A = 180^o - \angle \frac{1}{2}A$

This shows that E is on the circle with the center A and which goes through C_1 and $ß_2$.

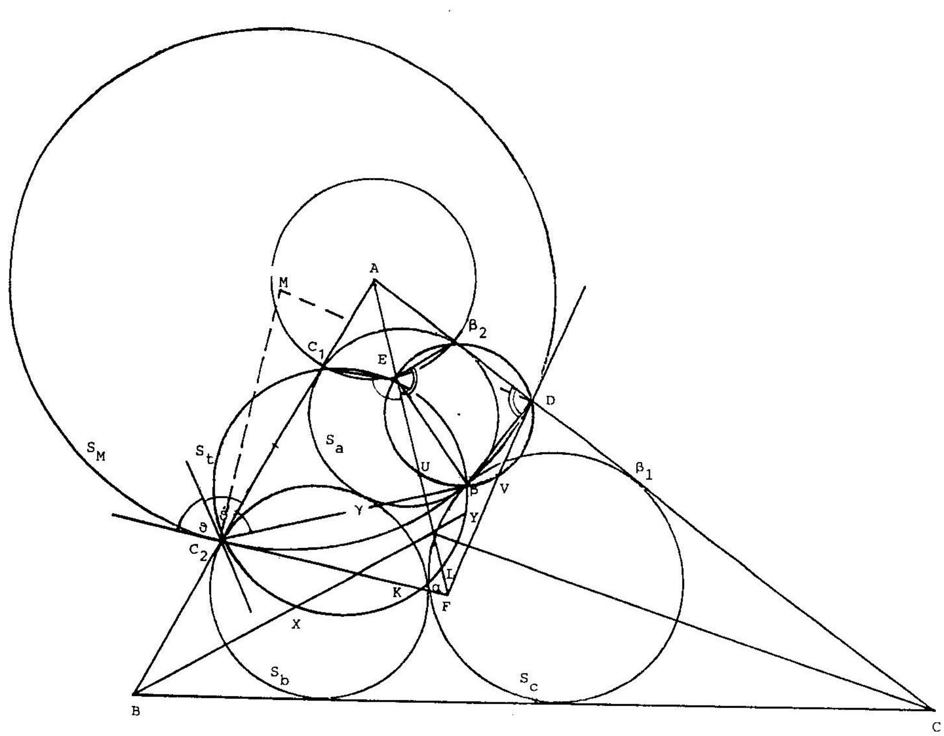

Fig.9

Fig.9 Consider the circle S_t.

$C_1C_2 = EL$ Because $AC_1 = AE$

We prove now that $C_2K = C_1C_2$

To show that these two segments of the circle S_t are equal, we will prove that the angles they make with the tangent to S_t at C_2 are equal. That is

? = ?'

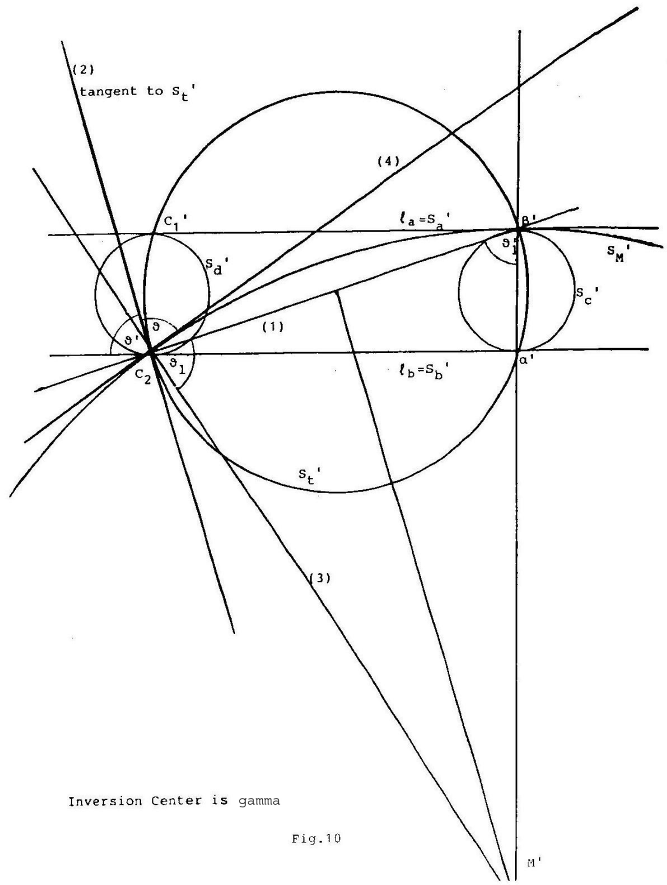

Fig.10 We invert the system with respect to the inversion center ?.

The circle S_a becomes a line l_a.

The circle S_b becomes a line l_b.

These two lines are parallel because the only common point ? of the two circles S_a and S_b goes to infinity.

The line C_1C_2 becomes a circle.

Thus S_b and C_1C_2 exchange roles under inversion.

Before the inversion the line C_1C_2 is tangent to the circle S_b.

After the inversion the circle $C'_1C'_2$ is tangent to the line l_b.

Also, the line C_1C_2 was tangent to the circles S_a, and S_b before the inversion..

After the inversion the circle $C'_1C'_2$ becomes tangent to the parallel lines l_a, and l_b.

The circle S_M through the points C_2 and ß becomes a circle S'_M through the points C'_2, an ß'.

The circle S_t through the points C_1,C_2 and ß becomes a circle S'_t through the points C'_1, C'_2, and ß'.

The circle S_C which was tangent to the circles S_a, and S_b before the inversion becomes a circle S'_C tangent to the parallel lines l_a , and l_b after the inversion. Consider now the inverted drawing.

Because l_a is tangent to S'_M at ß', a perpendicular line to this tangent goes through the center M' of S'_M.

Take the tangent line (4) to S'_M at C'_2 and draw a perpendicular line (3) to this tangent. This perpendicular line goes through the center M' of S'_M.

Note also that C'_2ß' is a diameter of S'_t.

In the uninverted drawing ? is the angle between C_2F (tangent to S_M at C_2) and the tangent to S_t at C_2.

When this is inverted ? is the angle between the line (4) (tangent to S'_M at C'_2) and the line (2) (tangent to S'_t at C'_2).

The line (1) C'_2ß' is $\perp$ to the line (2), because a radius of S'_t is $\perp$ to the tangent of S'_t.

Also the line (3) C'_2M' is $\perp$ to the line (4), because a radius of S'_M is $\perp$ to the tangent of S'_M.

Hence ? = $?_1$

Next in the uninverted drawing , ?' is the angle between the tangent to S_t at C_2 and C_1C_2 which is the tangent line to the circle S_b.

When this is inverted, the tangent to S_t at C_2 becomes the tangent to S'_t at C'_2 which is the line (2).

C_1C_2, which was the tangent line to the circle S_b becomes the circle $C'_1C'_2$ tangent to the line l_b. Thus ?' is the angle between the lines (2) and l_b .

Now M'ß' $\perp$ l_b

Line (1) $\perp$ Line (2)

$?_1$ = $?'_1$

But we already found that ? = $?_1$.

Thus ? = $?_1$ = $?'_1$ = ?'

? = ?'

since the inversion does not change angles at intersections of curves, we find that also in the uninverted drawing

? = ?' .

this proves that $C_2K = C_2C_1$

Because these segments are the corresponding segments to the tangent line to S_t at C_2 with equal angles.

We now observe that the circles S_t and S_u play similar roles with respect to circle S_M and the point F. We proved that C_2C_1=C_2K where C_2K is on the tangent from F to CM. Exactly the same proof can be given to show that Dß$_2$=DV where DV is on the tangent from F to C_M. consequently EU=DV.

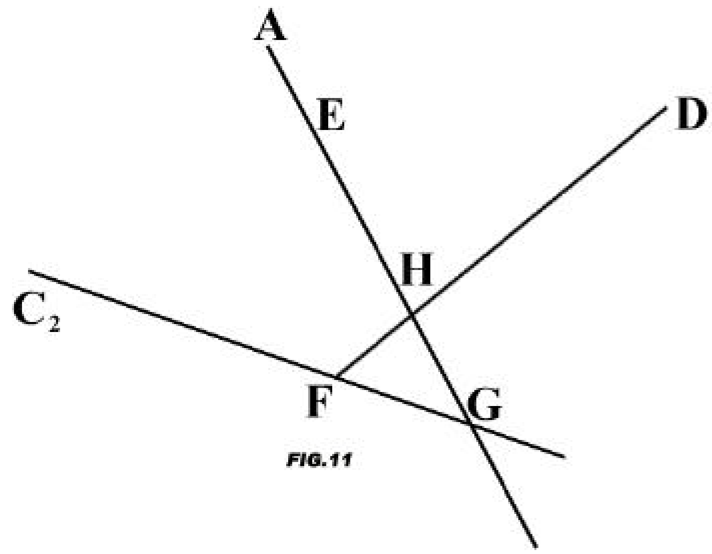

FIG.11

Fig.11. There remains one more point to be proven. Namely, the line AE goes through F.
This we prove now:
Suppose AE does not go through F. Let us call the intersection points of AE with the tangent line C_2F G and with the tangent line FD H.
It then follows:

(1) $FC_2 = FD$ (Tangents to the same circle S_M)
(2) $GC_2 = GE$ (Because of the equality of the segments $C_2K = EL$)
(3) $HD = HE$ (Because of the equality of the segments $EU = DV$)

Subtracting (1)-(2) $GC_2 - FC_2 = FG = FD\text{-}GE = FH+HD\text{-}(GH+HE)$

$$FG = FH-GH+HD-HE$$

$$FG+GH = FH$$

This tells us that the sum of the two sides of a triangle equals its third side.
Hence F, G and H are one and the

This means that E is on the bisector AF.

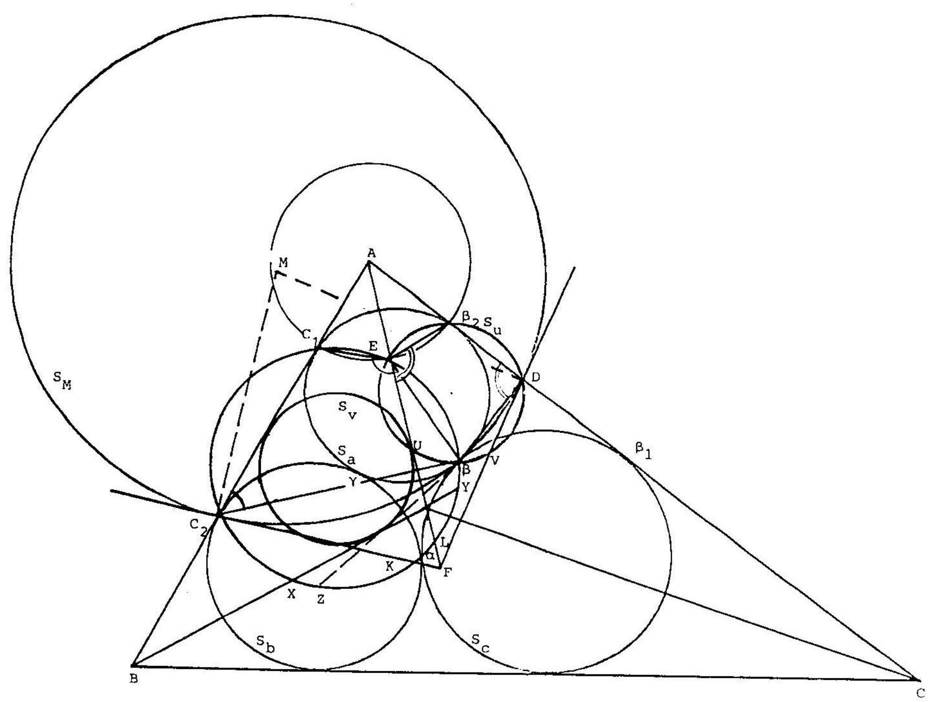

Fig.12

Fig.12. Now we show that the segment XY of the bisector of the angle $\angle B$ is also equal to the segments we discussed. Namely:

$$XY = C_2C_1 = C_2K = EL$$

The reason for this is as follows:

The segment C_2C_1 plays similar roles with respect to the two sides of our triangle (of A and B).

We focused on the circles S_a and S_u on the A side. We thus found that

$$AC_1 = AE \quad \text{and} \quad C_2C_1 = EL.$$

Had we constructed the analogue of the circle S_u on the B corner side, we would have found the counterpart of

$$C_2C_1 = EL \text{ which is } C_2C_1 = XY.$$

Next, construct the tangent line to S_a and S_c at ß.

This gives us the segment ßZ, and

$$ßZ = C_1C_2.$$

The reason for this is as follows:

Consider the circles S_t and S_a. they intersect at C_1 and ß. If we construct at C_1 and ß tangents to S_a they are symmetric with respect to the line connecting the centers of S_t, and S_a. Therefore

$$ßZ = C_1C_2.$$

Thus we have the following segments of S_t which are all equal.

$$C_1C_2 = EL = C_2K = XY = ßZ$$

Consequently we can draw a circle which is tangent to all those segments. Let us call this circle S_v. Obviously S_v has the same center as the circle S_t. this circle is also seen to be the in-circle of the triangle ABO where O is the intersection point of the bisectors of the triangle ABC.

Moreover the tangent to the circles S_a, and S_c at ß is also tangent to this circle S_v.

Finally this very tangent is also tangent to the in-circle of the triangle OBC, which we will call S_w. The reason for this is that the roles of S_v and S_w with respect to the circles S_a, and S_c are similar. Had we focused on S_w instead of on S_v we would find that the tangent line to S_a and S_c at ß is also tangent to the circle S_w. This finally proves Steiner's conjecture.

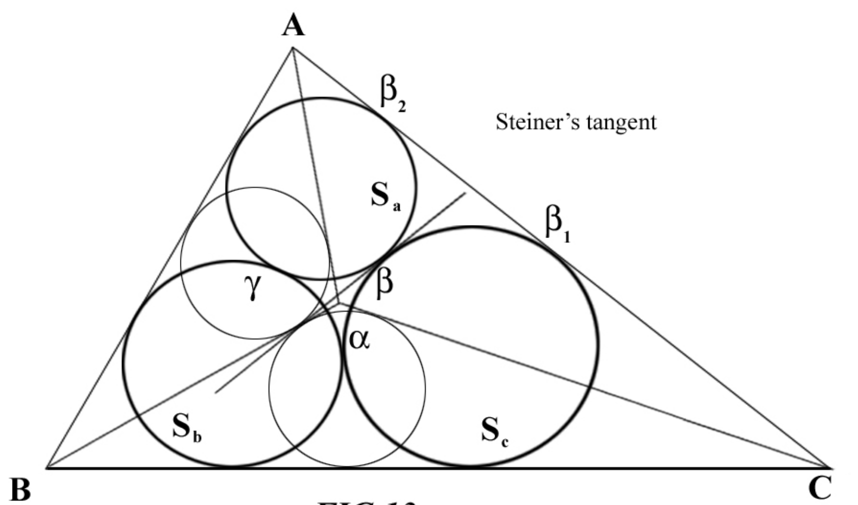

FIG.13

Fig.13 Steiner's Conjecture:

Suppose the problem of finding three Malfatti circles is solved. Then the tangent line to two of these three circles , say S_a and S_c at their contact point ß is tangent to both of the circles S_v, and S_w inscribed into the triangles ABO and OBC where O is the intersection point of the bisectors of the angles of the given triangle ABC.

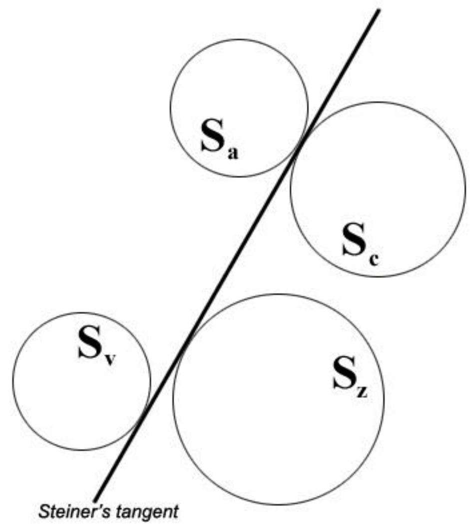

FIG.14

Magnified picture of Steiner's tangent

Fig.14 Since Steiner's line is so difficult to recognize in an actual triangle ,we give here an exaggerated picture of it. Here S_a and S_c are two of the Malfatti-Steiner circles. The other circles are two of the circles inscribed into the triangles formed by one side of the triangle and two of the bisectors.

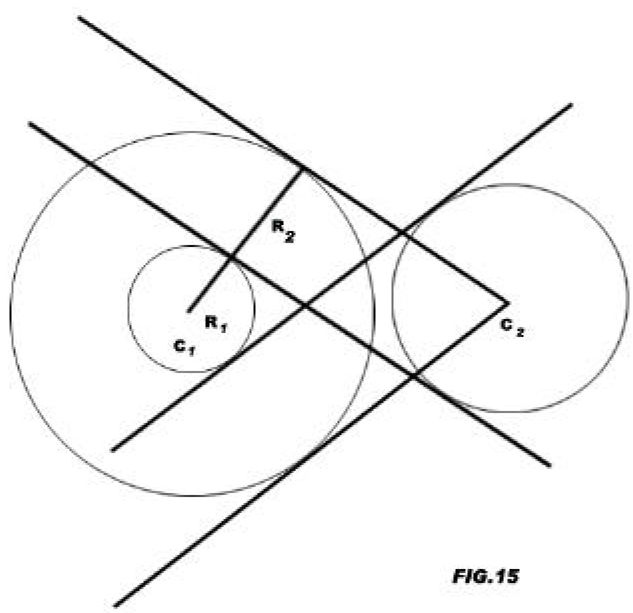

FIG.15

Fig.15 Construction:

Constructing the three bisectors AO, BO, and CO of the triangle ABC , three triangles are found. Construct the in-circle S_v of the triangle ABO. Construct the in-circle S_w of the triangle OBC.

Construct one of the common inner tangents of the circles S_v and S_w. Call this line the Steiner line.
Steiner line is also tangent to one of the three Malfatti circles. Thus the problem is to construct the circle which touches two sides, say AB and CA of the triangle ABC and the Steiner line.
To construct the common inner tangent of two given circles with centers C_1, and C_2 and radii R_1, and R_2 we draw around the circle C_1 a circle with radius $R_1 + R_2$.

We next draw a tangent from the point C_2 to this circle . Finally we draw a parallel line to this tangent at a distance R_2 on the circle's side. This gives the common inner tangent of the given circle.

A C K N O W L E D G M E N T S

I dedicate this paper to my late friend Umit Camat, who gave me Julius Petersen's book as a present. Without him I would not have the motivation for working on this problem.
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