RANDOM GRAPHS AND A DISCRETE APPROACH TO QUANTUM GRAVITY

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ABSTRACT. We study a collection of Markov chains with values in the collection of partial orderings of the natural numbers. These chains arise naturally in the context of discrete theories of quantum gravity and include the well-studied example of "transitive percolation." Using transitive percolation as a case study, we present recent results and sketch corresponding developments in the associated context of quantum gravity.

1. INTRODUCTION

The study of random graphs was formalized by Erdös and Renyi in the early 1960's. This expository note concerns a model of random graphs investigated twenty years ago by Barak and Erdös. This model, called transitive percolation in the physics community, has recently been of interest in developing discrete models for quantum gravity.

Transitive percolation, the standard model for producing random partial orders on the natural numbers, is easy to describe: Fix $0 < p < 1$ and consider the following inductive scheme:

(1) Let A_0 be the directed graph consisting of a single labeled vertex v_0.
(2) Given a directed graph on n vertices labeled $v_0,\ v_1, \ldots, v_{n-1}$ with edges consistent with the natural ordering on the vertex labelings, introduce a new vertex (labeled n) and with probability p introduce an edge directed from the new vertex to each of the existing vertices, the addition of each edge being independent of all other edges.
(3) Take the transitive closure of the partial order obtained once the random edges are determined.

A great deal is known about transitive percolation: Barak and Erdòs [BE] investigated the width of random graphs generated via transitive

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percolation. Treating an edge between vertices v_i and v_j as defining an ordering relation on the vertex lables, Albert and Friese established concentration phenomena for the height and set-up number of associated random order in the case $p = \frac{1}{2}$ [AF]. Extending work on random orders, Bollobás and Brightwell gave precise estimates for the width of a random graph [BB1] and for the corresponding dimension [BB2] (see [JLR] for an extensive introduction to random graphs and transitive percolation).

Using transitive percolation to generate random partial orders on the natural numbers, Alon, Bollobas, Brightwell and Janson [ABBJ] study the properties of *posts* (ie nodes which are connected to all other nodes under the partial order). Since the appearance of [ABBJ] a great deal of work has gone into understanding the properties of posts (cf [BB3], [KP] and references therein). In addition, much work has focussed on applications outside of mathematics which involve transitive percolation and post formation. One such application involves cosmology.

For those working in cosomology, transitive percolation provides the simplest example from a collection of toy models which serve as classical precursors for a discrete theory of quantum gravity. This theory, called the causet model of quantum gravity, has been developed by Sorkin and his collaborators (cf [BLMS], [BDGHS1], [BDGHS2], [DS], [MORS], [RS], [S1], [S2] and references therein). At its most primitive, the causal set idea posits that the structure of space and time is that of a partially ordered locally finite set (causet). The dynamics for the theory are random and the familiar objects of associated to general relativity (in particular a manifold with Lorentz metric) appear as good approximations to causets (not vice-versa). The properties of random graphs cited in the above literature take on familiar physical significance (eg the length of the longest chain between two causally related elements gives the proper time, etc).

The primary objective of this note is to provide an introduction to causet dynamics through the example of transitive percolation. As such is the case, we focus on computing simple examples which provide motivation for the definitions of the fundamental objects which describe causet dynamics, relegating proofs of the technical results to the references. Of the many references dedicated to causet dynamics, [RS] and [S3] are recommended reading for those interested in a more detailed introduction to causets and transitive percolation.

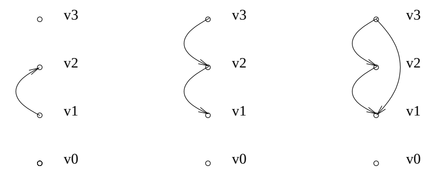

FIGURE 1. Examples of graphs

2. Foundations and General Covariance

From our description of transitive percolation, the process takes values in the collection of directed graphs with vertices indexed by the natural numbers, where the edge directions respect the vertex labels. We will call the collection of such graphs on n vertices *n-allowable.* We will denote the collection of n-allowable graphs by $\tilde{\mathcal{C}}_n$ and we will denote by $\tilde{\mathcal{C}}$ the state space of our process:

$$\tilde{\mathcal{C}} = \cup \tilde{\mathcal{C}}_n.$$

We will always draw graphs with labeled nodes arranged vertically. For example, Figure 1 features three directed graphs, exactly one of which is allowable (the first graph in the picture does not respect vertex labelling, while the second is not transitively closed). In what follows, vertex labels will be understood to be increasing with height and thus edge directions will be understood to be directed downward.

From our description of transitive percolation it is clear that the discrete transitions which occur involve the introduction of a single new vertex and edges which connect the new vertex to previously introduced vertices - and nothing more. In particular, given any allowable graph $C \in \tilde{\mathcal{C}}_n$, there is a finite collection of n-allowable graphs to which C can evolve with nonzero probability under transitive percolation. Given $C \in \tilde{\mathcal{C}}_n$ we will refer to the finite collection of n-allowable graphs to which C can evolve with positive probability as the *family of* C, denoted $F(C)$. We will refer to the transitions $C \to D \in F(C)$ which occur with positive probability under transitive percolation as *allowable.*

Figure 2 provides a "seed" (the allowable graph C) and all allowable transitions (via transitive percolation) from C to another allowable graph D. As indicated in the figure, there are exactly six such transitions for the given graph C; the corresponding probabilities are recorded in Table 1.

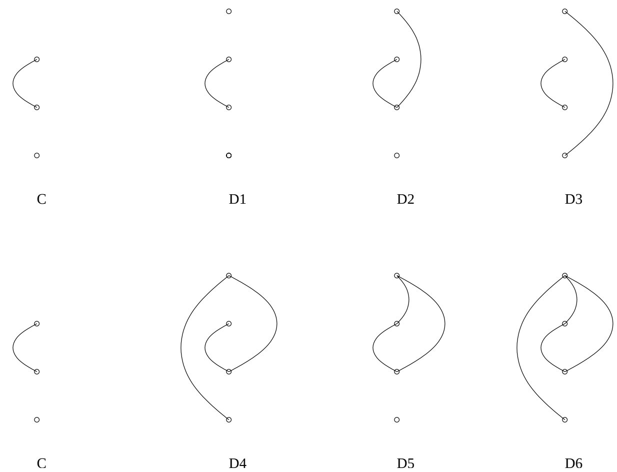

FIGURE 2. An allowable graph and its family

$\text{Prob}(C \to D_1) = (1-p)^3$		$\text{Prob}(C \to D_4) = (1-p)p^2$
$\text{Prob}(C \to D_2) = (1-p)^2 p$		$\text{Prob}(C \to D_5) = (1-p)p$
$\text{Prob}(C \to D_3) = (1-p)^2 p$		$\text{Prob}(C \to D_6) = p^2$

Table 1: Transition probabilities for figure 2

We will interpret the existence of an edge between vertices as the statement "in the causal past of." We will call elements in the causal past of a vertex v_n the *precursors of* v_n. If the only path between a pair of vertices $v_l,\ v_n$ with $l < n$ is a single edge, we will say that v_l is in the immediate causal past of v_n. If v_l is in the immediate causal past of v_n we will say that v_l *is maximal* in the past of v_n. Given an allowable transition $\tilde{\mathcal{C}}_n \ni C \to D \in F(C)$, we set

$$r = \text{number of elements in the causal past of } v_n \tag{2.1}$$

$$m = \text{number of maximal elements associated to } v_n.\ . \tag{2.2}$$

Table 2 records the values of r and m for each of the transitions given in Figure 2:

	D1	D2	D3	D4	D5	D6
r	0	1	1	2	2	3
m	0	1	1	2	1	2

Table 2: Precursors and maximal elements for Figure 2

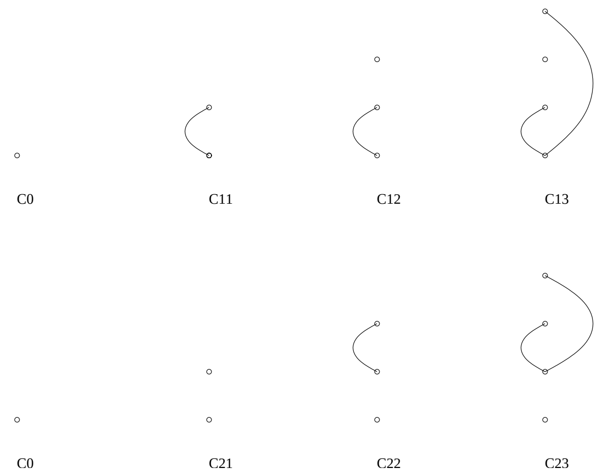

FIGURE 3. Two examples of evolution under transitive percolation

It is easy to check that for any of the transitions pictured in Figure 2, we have

$$\mathrm{Prob}(C \to D_i) \;=\; p^m(1-p)^{3-r}. \tag{2.3}$$

In fact, the analog of (2.3) holds for any allowable transition under transitive percolation: If $C \in \tilde{\mathcal{C}}_{n-1}$ and $C \to D \in F(C)$ is an allowable transition, then

$$\mathrm{Prob}(C \to D) \;=\; p^m(1-p)^{n-r} \tag{2.4}$$

where r and m are given as in (2.1) and (2.2), respectively. To see that the computation is correct, note that to completely specify the transition, we need only specify the number of edges drawn to vertices in the immediate past, and the size of the past (ie the size of the transitive closure of the graph obtained by adding edges between the new vertex and the chosen maximal elements).

Formula (2.4) makes it easy to compute transition probabilities for transitive percolation and thus to study properties of the trajectories associated to transitive percolation as a process taking values in the collection of random graphs. Again, we begin with an example.

Figure 3 contains possible initial segments for transitive percolation trajectories. Using (2.4) we can compute the probability that any given trajectory begins as pictured:

$$\begin{aligned} \mathrm{Prob}(C_0 \to C_{11}) &= p \\ \mathrm{Prob}(C_{11} \to C_{12}) &= (1-p)^2 \\ \mathrm{Prob}(C_{12} \to C_{13}) &= (1-p)^2 p \end{aligned}$$

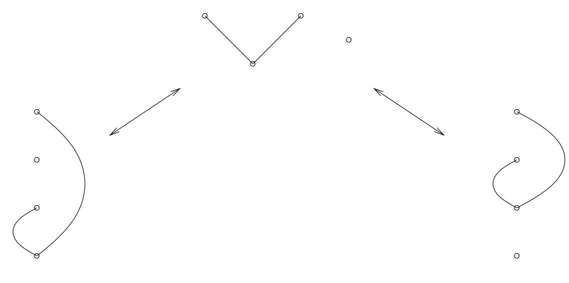

FIGURE 4. Isomorphism classes of allowable graphs

and

$$\begin{aligned}\mathrm{Prob}(C_0 \to C_{21}) &= (1-p)\\ \mathrm{Prob}(C_{21} \to C_{22}) &= (1-p)p\\ \mathrm{Prob}(C_{22} \to C_{23}) &= (1-p)^2 p.\end{aligned}$$

We conclude that

$$\mathrm{Prob}(C_0 \to C_{11} \to C_{12} \to C_{13}) \quad = \quad \mathrm{Prob}(C_0 \to C_{21} \to C_{22} \to C_{23}).$$

It isn't hard to believe that the example developed in Figure 3 and the computation above represent a property of transitive percolation which is reflected by other sequences of transitions. To properly quantify this phenomena, we begin by noting that if we formally "erase labels" from the diagrams appearing in Figure 3, the states corresponding to C_{13} and C_{23} are indistinguisable; Figure 4 encodes this observation.

Definition 2.1. We say that two allowable graphs are isomorphic if there is a bijection which maps vertices to vertices and preserves edge structure. We will call the collection of equivalence classes of allowable graphs *causets*. We denote the collection of equivalence classes of n-allowable graphs by $\mathcal{C}_n$. We denote the collection of all causets by $\mathcal{C}$:

$$\mathcal{C} = \cup \mathcal{C}_n.$$

There is a natural physical interpretation we can associate to the study of "allowable graphs with labels erased:" By introducing a labeling, we introduced an element of "temperal gauge." Thus, by erasing the labelings, we are attempting to study those properties which do not depend on temporal gauge, ie those that are covariant (for more on this theme see [BDGHS1] and references therein).

Given $C \in \mathcal{C}_n$, let $\tilde{C}$ be any allowable graph representing C. We can define the family of C as those causets which arise as equivalence classes of elements of $F(\tilde{C})$. This said, we can broaden the class of processes which we will consider:

Definition 2.2. We say that a Markov chain M with state space $\mathcal{C}$ belongs to the collection $\mathcal{M}_{\text{gc}}$ if the transition probabilities of M satisfy:

(1) (Locality) Given $C \in \mathcal{C}_n$, let $\text{Prob}(C \to D)$ denote the transition probability corresponding to an evolution from causet C to causet D. Then $\text{Prob}(C \to D) = 0$ if $D \notin F(C)$ and $\sum_{D \in F(C)} \text{Prob}(C \to D) = 1$
(2) (General Covariance) Let $C \in \mathcal{C}_n$. Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are two paths from the trivial causet consisting of a single point to C and write $\mathcal{P}_i = \{l_{i1}, \ldots, l_{in}\}$ where the l_{ij} are the links defining the path $\mathcal{P}_i$. Then

$$\prod_{k=1}^{n} \text{Prob}(l_{1k}) = \prod_{k=1}^{n} \text{Prob}(l_{2k}).$$

It is clear from our description that transitive percolation defines a Markov process taking values in causets which satifies locality. Our computations suggest that it also satisfies general covariance. Indeed, this is a theorem of Rideout and Sorkin:

Theorem 2.3. [RS] *Transitive percolation satisfies general covariance.*

We will henceforth consider transitive percolation with state space $\mathcal{C}$.

3. Bell Causality

In Figure 5 we picture two possible transitions between allowable graphs.

In Figure 6 we consider two causet transitions with initial causet representing the isomorphism class of the allowable graphs appearing in Figure 5 (we have omitted edges which clearly must be present given that all graphs must be transitively closed). Note that each transition involves drawing connections from a new vertex to a number of the vertices in C_0. We have circled the vertices of C_0 which are involved in each transition. For each such transition, the circled vertices form a causet; the *the precursor causet of each transition.*

Using the information represented in Figure 5 and Figure 6 we can prescribe an algorithm to construct "induced" causet transitions:

(1) Construct the union of the causets involved in each of the transitions. This is a causet.

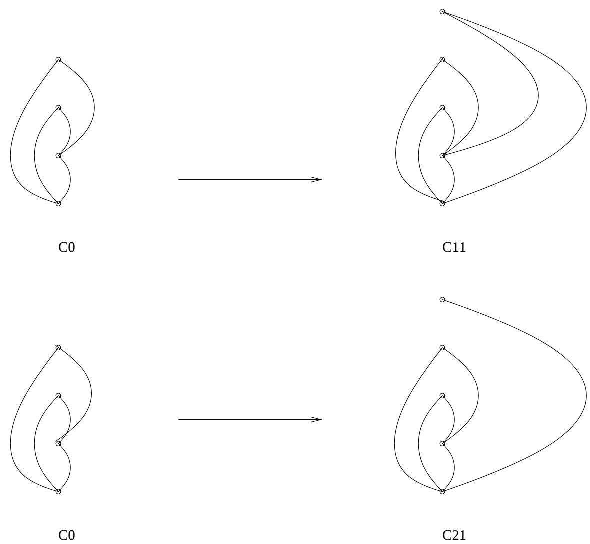

FIGURE 5. Causet transitions

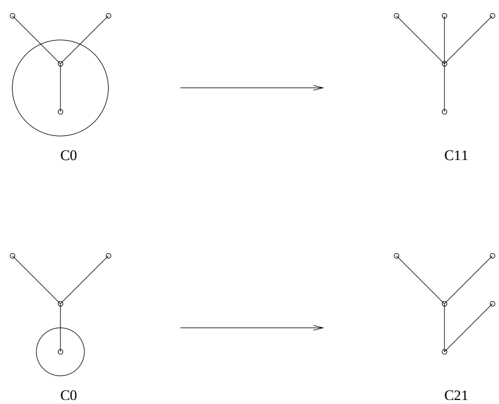

FIGURE 6. Causet transitions

(2) Beginning with the union of causets involved in each transition, add a vertex and edges as was done in the original transitions. This produces a pair of transitions: the induced causet transitions.

For the transitions given in Figure 5 and Figure 6 the resulting induced transitions are pictured in Figure 7 (cf [RS]):

For the transitions described in Figure 6 and Figure 7 we can compute the associated transition probabilities:

$$\begin{aligned} \text{Prob}(C_0 \to C_{11}) &= p(1-p)^2 \\ \text{Prob}(C_0 \to C_{21}) &= p(1-p)^3 \end{aligned}$$

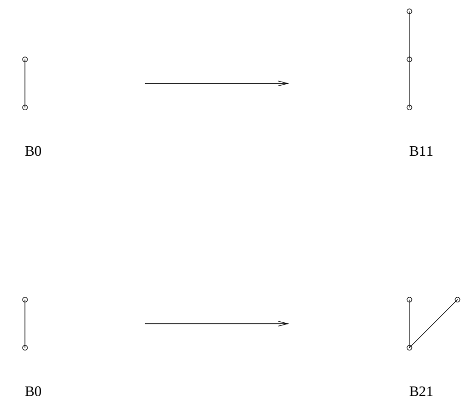

Figure 7. Induced causet transitions from figure 3

and

$$\begin{aligned} \mathrm{Prob}(B_0 \to B_{11}) &= p \\ \mathrm{Prob}(B_0 \to B_{21}) &= p(1-p). \end{aligned}$$

We note

$$\frac{\mathrm{Prob}(B_0 \to B_{11})}{\mathrm{Prob}(B_0 \to B_{21})} = \frac{\mathrm{Prob}(C_0 \to C_{11})}{\mathrm{Prob}(C_0 \to C_{21})}. \tag{3.1}$$

Definition 3.1. Let M be a Markov chain taking values in the collection of causets. Let $C_0 \to C_{11}$ and $C_0 \to C_{21}$ be transitions involving the introduction of one vertex v_n. Let

$$B_0 = \mathrm{past}_{C_{11}}(v_n) \cup \mathrm{past}_{C_{21}}(v_n) \tag{3.2}$$

and let $B_0 \to B_{11}$, and $B_0 \to B_{21}$ be the induced transitions obtained by adding a vertex to the vertex set of B_0 and adding edges determined by the transitions $C_0 \to C_{11}$ and $C_0 \to C_{21}$, respectively. If

$$\frac{\mathrm{Prob}(B_0 \to B_{11})}{\mathrm{Prob}(B_0 \to B_{21})} = \frac{\mathrm{Prob}(C_0 \to C_{11})}{\mathrm{Prob}(C_0 \to C_{21})} \tag{3.3}$$

for any allowable one step transitions $C_0 \to C_{11}$ and $C_0 \to C_{21}$, we say that M satisfies *Bell Causality.*

With this definition, we can establish (as did Rideout and Sorkin):

Theorem 3.2. [RS] *Transitive percolation satisfies Bell causality.*

4. Preliminary Classification

Definition 4.1. We say that a Markov chain M taking values in the collection of causets and satisfying locality (Definition 2.2), general covariance (Definition 2.2) and Bell causality (Definition 3.1) is admissable.

We have seen that transitive percolation is admissable. Our present goal is to identify other examples. A central role in our investigation will be played by transitions between causets with no relations:

Definition 4.2. Let M be an admissable Markov chain. Let $A_n \in \mathcal{C}_n$ be the causet having $n+1$ vertices and no edges. The defining probabilities associated to M are given by

$$q_n = \mathrm{Prob}(A_n \to A_{n+1}). \tag{4.1}$$

We say that M is *generic* if the defining probabilities for M satisfy $q_n > 0$ for all n.

We can compute the defining probabilities for transitive percolation:

$$q_n = (1-p)^n \tag{4.2}$$

and thus transitive percolation is generic.

It turns out that defining probabilites completely determine the transition probabilities of the corresponding chain [RS], [AM1].

For the purpose of computation, it's useful to introduce a second description of transitive percolation.

Definition 4.3. Let M be an admissable chain with defining probabilities q_n. The coupling constants associated to M are defined by

$$t_m = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{1}{q_k}.$$

Using the binomial theorem, we can give a closed form expression of the coupling constants for transitive percolation in terms of defining parameter p :

$$t_n = \left(\frac{p}{1-p}\right)^n.$$

Note that for transitive percolation we can use (4.3) to recover the defining probabilities in terms of the coupling constants [RS]:

$$\frac{1}{q_n} = \sum_{k=0}^{n} \binom{n}{k} t_n. \tag{4.3}$$

In fact, the identity (4.3) holds between the coupling constants and defining probabilities of any admissable generic Markov chain:

Theorem 4.4. [RS]*Let M be an admissable generic Markov chain with defining probabilities q_n and coupling constants t_n. Then*

$$t_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{q_k}$$
$$\frac{1}{q_n} = \sum_{k=0}^{n} \binom{n}{k} t_n.$$

That (4.3) holds for generic admissable chains can be checked directly via computation. The identity can also be checked by giving a second description of transitive percolation (as well as any other generic admissable chain.

Since defining probabilities determine a generic admissable chain and coupling constants determine the corresponding defining probabilities, it should be possible to classify admissable generic chains via coupling constants. This is indeed the case:

Theorem 4.5. [RS] *Suppose that t_n is a sequence of real numbers and suppose that the sequence q_n is defined by (4.3). Then the sequence q_n form the defining probabilities of an admissable Markov chain if and only if the sequence t_n is nonnegative.*

5. Cosmic Renormalizability

Definition 5.1. Let S be the collection of nonegative sequences with positive first term. The renormalization map $\mathcal{R} : S \to S$ is defined by

$$(\mathcal{R}(\{t_k\}))_n = \frac{t_{n+1} + t_n}{t_1 + t_0}. \tag{5.1}$$

We say that a sequence $\{t_n\}$ is stable under renormalization if

$$\{t_n\} \in \cap_{k=1}^{\infty} \mathcal{R}^k(S).$$

We say that a Markov chain is renormalizable if its associated collection of coupling constants is renormalizable. We say that a generic admissable chain is a gcd chain if it is renormalizable.

We can compute the action of the renormalization map on the coupling constants associated to transitive percolation: Setting $\delta = \frac{p}{1-p}$, we have

$$(\mathcal{R}(\{t_k\}))_n = \frac{\delta^{n+1} + \delta^n}{\delta + 1} \tag{5.2}$$
$$= \delta^n. \tag{5.3}$$

That is,

Theorem 5.2. [MORS] *Transitive percolation defines a gcd chain.*

There are good physical reasons for introducing the notion of cosmic renormalizability. Briefly, the renormalization operator models the effect of cycles of cosmic expansion and contraction (see [MORS] and [S1] for more details).

There is a sense in which every gcd chain is composed of transitive percolations. To make this more precise, suppose that $H_\delta(x)$ is a Heaviside function with jump at δ :

$$H_\delta(x) \quad = \quad \begin{cases} 0 & \text{if } x \leq \delta \\ 1 & \text{if } x > \delta. \end{cases}$$

We can write the coupling constants of transitive percolation as moments of a Stieltjes measure:

$$t_k \quad = \quad \int s^k dH_\delta(s). \tag{5.4}$$

We have

Theorem 5.3. [AM1] *Suppose that M is an admissable Markov chain which is generic. Let $\{t_n\}$ be the coupling constants associated to M. Then M is a gcd chain if and only if there is a nondecreasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ such that*

$$t_n \quad = \quad \int_{\mathbb{R}^+} s^n d\alpha(s). \tag{5.5}$$

The proof of this theorem involves the classical moment problem. The theorem itself gives a map between solutions of the classical moment problem and gcd chains.

6. Corollaries

In addition to providing new examples gcd chains, Theorem 5.3 leads to a collection of both mathematical and physical results. We focus our attention on one such class of results: the formation of posts.

Let M be a gcd chain. The trajectories associated to M are paths in the collection of admissable graphs: each such path consists of a sequence $\{C_n\}$, $C_n \in \mathcal{C}_n$ where $C_{n+1} \in F(C_n)$ for every n. By construction, each such path defines a partial order on the natural numbers: given a path, $\{C_n\}$ and two natural numbers n and m, we say that m comes before n if there is an edge between n and m in C_n.

Definition 6.1. Let $\mathcal{P}$ be a partial order on the natural numbers. A natural number n is a post with respect to $\mathcal{P}$ if n is $\mathcal{P}$-ordered with respect to every other natural number.

The notion of a post has proven to be very useful: posts appear in a applications in a number of fields including mathematics, physics and computer science (cf [BB3] and references therein). We focus on their applications in cosmology.

From the point of our discrete model, post formation corresponds to a collapse of the universe, followed by a re-expansion. For this reason it is important to know whether they occur. For the case of transitive percolation, this is a result af Alon, Bollobás, Brightwell and Jantzen:

Theorem 6.2. [ABBJ] *Under the dynamics associated to transitive percolation, posts occur infinitely often, almost surely.*

Using Theorem 5.3, we can extend this result to a large collection of gcd chains:

Theorem 6.3. [AM2] *Let M be a gcd chain represented by a nondecreasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}$. Suppose that the support of the measure $d\alpha$ is compact and that the supremum of the support is isolated. Then under the dynamics associated to M, posts occur infinitely often, almost surely.*

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