# Let's get into the game spirits 

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## 0. Introduction

Games have challenged and entertained mankind throughout its history, as is witnessed by the popularity of chess, "Go", etc.. The main appeal of these games is certainly their complexity and the lack of complete winning strategy (in the mathematical sense). On the other hand, once the winning strategy is known, many simpler games (e.g. Nim) loose their appeal as competitive games, but gain popularity as good recreational topics in mathematical education. The search for such winning strategies have also attracted more and more research interests in the recent decades.

In this talk, I will present some simple combinatorial games and their winning strategies, and hope to illustrate the important role they should have in our classrooms.

## 1. Some well-known examples

We start with a "Problem of the Week" I posed at Polk Community College.
A "Problem of the Week".
To part with Year 2002 and welcome Year 2003, Alice and Bob play a 'domino' game on a gigantic 2002-by-2003 rectangular chess board. The players alternate moves, with
Alice going first. At each move, a player covers two adjacent unit squares by a domino. The player who cannot find two adjacent empty squares to cover, loses. Who can force a win, and how?

## Solution.

Alice can force a win, by a strategy of "symmetry".
Games are often good competition problems, as seen in the recent 2002 Putnam paper.

## Putnam problems.

A4. In Determinant Tic-Tac-Toe, player 1 enters a 1 in an empty 3-by-3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3-by-3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?
B2. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:
Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.
Show that the player who signs first will always win by playing as well as possible.
We use another simple game to explain what we mean mathematically by a winning strategy.

## Bachet's game.

Initially, there are 100 checkers on the table. Two players take turn to remove at least 1 and at most 7 checkers each time from the table. The last player who can remove any checker wins the game. Which player can force a win, and how?
Solution.
The first player can always win.
Partition the positions into two complementary sets $A$ and $B$, where $A=\{n: n=8 k\}$.
Observe:

1. Any move from a position in $A$ must end up with a position in $B$; and
2. Any position in $B$ can be moved to a position in $A$.
3. The starting position 100 is in $B$ and the ending position 0 is in $A$.

Finally, we mention the best-known game of Nim, the winning strategy of which was described by C. L. Bouton (1902), R. P. Sprague (1935-36), and P. M. Grundy (1939) independently.
The game of Nim.
The game involves several piles of counters. On each player's turn, she selects one of the remaining piles and removes any number ( $>0$ ) of counters from it. The player who makes the last move wins.

## Solution.

See, for example, [1].

## 2. The game of Blocking Nim

Problem 714. (CMJ, [2002, 414-415]) Proposed by Arthur L. Holshouser, Charlotte, NC and Harold B. Reiter, University of North Carolina, NC
The game of "Blocking Nim" proceeds in exactly the same way as ordinary Nim, except that before a given player takes his turn, his opponent is allowed to announce a "block", i. e., for each pile of counters, he has the option of specifying a positive number of counters which may not be removed from that pile. Thus, when play begins, $P_{1}$ announces a block, and $P_{2}$ takes a turn that is consistent with the announced block. $P_{2}$ then announces a block, and $P_{1}$ takes turn that is consistent with this block, and so on. The winner is the one who either removes the last of the counters or who leaves the opponent unable to remove a counter from any of the remaining piles.

## Solution.

For $1 \leq i \leq m$, let $a_{i}$ be the number of counters in the $i$ th pile and $b_{i}$ the blocking number for the $i$ th pile. Represent $a_{i}$ in base 2 as $a_{i}=a_{i, n} 2^{n}+a_{i, n-1} 2^{n-1}+\ldots+a_{i, 1} 2$

$$
+a_{i, 0} . \text { Let } A=\left\{\left(\begin{array}{cc}
a_{1} & b_{1} \\
a_{2} & b_{2} \\
\ldots & \ldots \\
a_{m} & b_{m}
\end{array}\right): S_{j}=\sum_{i=1}^{m} a_{i, j} \equiv 0(\bmod 2) \text { for each } 1 \leq j \leq n ; \text { and } b_{i}=1\right.
$$

whenever $\left.a_{i, 0}=1\right\}$. Let $B$ be the set of positions not in $A$.
Observe:

1. The final positions of the game are in $A$.
2. Any move from a position in $A$ must make $S_{j}$ odd for some $j \geq 1$, and thus must end up with a position in $B$.
3. On the other hand, suppose $p$ is a position in $B$.

Case I. If $p$ has an odd $S_{j}$ for some $j \geq 1$, then we always have an $a_{i}$ and two choices of positive integers $c$ and $c+1$ that we can remove from $a_{i}$ to make $S_{j}$ even for all $j \geq 1$. Since one of $c$ or $c+1$ must differ from $b_{i}$, we see that $p$ can be moved to a position in A.

Case II. If $p$ has even $S_{j}$ for all $j \geq 1$, then there exists $i$ such that $a_{i, 0}=1$ but $b_{i} \neq 1$. By removing 1 from $a_{i}$ and announcing the correct blocking, we move $p$ again into a position in $A$.

Further question. Proposed by S. C. Locke, Florida Atlantic University, FL What if the block of each turn is on one pile only?

## 3. Sprague-Grundy function

The winning strategies mentioned above often seem "rabbit-out-of-the hat". But they are usually discovered by painstaking endgame analysis. One useful device for such analysis is the Sprague-Grundy function.

This is a function of the game positions, introduced by P. M. Grundy when analyzing the game of Nim. R. P. Sprague mentioned the idea of such a function earlier.

This function $g$ is defined recursively as follows:

1. A "terminal" position, that is a position from which no further move is possible, has the $g$-value 0 .
2. For a non-terminal position $p, g(p)=\min \left\{\mathbb{N}_{0} \backslash\{g(q): p\right.$ can reach $q$ in one move. $\left.\}\right\}$.

Thus, $g$ has the following properties:

1. No position with $g$-value $n$ can have a follower with the same $g$-value $n$.
2. A position with $g$-value $n$ has some follower with any smaller $g$-value, unless $n=0$.

Using this function $g$, we can partition the game positions into two disjoint sets $A=\{p$ : $g(p)=0\}$ and $B=\{p: g(p)>0\}$.

## 4. The game of Kim

We apply the Sprague-Grundy function to the analysis of a new game.
Problem 10951. (Monthly, [2002, 569]) Proposed by Sung Soo Kim, University of Guelph, Canada
A game starts with one stick of length 1 and four sticks of length 4 . The two players move alternately. A move consists of breaking a stick of length at least two into two sticks of shorter integer length or removing $n$ sticks of length $n$ for some $n \in\{1,2,3,4\}$. The player who makes the last move wins. Which player can force a win, and how?

## Dirty work on scratch papers.

Let $a_{1}, a_{2}, a_{3}, a_{4}$ be the numbers of sticks of lengths $1,2,3,4$ respectively. After some calculations, we obtain the following Grundy values $g\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ :
$a_{3}=0, a_{4}=0$

| $a_{1} \backslash a_{2}$ | 0 | 1 | 2 |
| :--- | :--- | :--- | :--- |
| 0 | 0 | 1 | 2 |
| 1 | 1 | 0 | 3 | | $a_{1} \backslash a_{2}$ | 0 | 1 | 2 |
| :--- | :--- | :--- | :--- | :--- |
| 0 | 1 | 2 | 0 |
| 1 | 0 | 3 | 1 |

$a_{3}=0, a_{4}=2$

| $a_{1} \backslash a_{2}$ | 0 | 1 | 2 |
| :--- | :--- | :--- | :--- |
| 0 | 2 | 0 | 1 |
| 1 | 3 | 1 | 0 |

$a_{3}=1, a_{4}=0$

| $a_{1} \backslash a_{2}$ | 0 | 1 | 2 |
| :--- | :--- | :--- | :--- |
| 0 | 1 | 0 | 2 |
| 1 | 0 | 1 | 3 |

$a_{3}=1, a_{4}=1$

| $a_{1} \backslash a_{2}$ | 0 | 1 | 2 |
| :--- | :--- | :--- | :--- |
| 0 | 0 | 2 | 1 |
| 1 | 1 | 3 | 0 |

$a_{3}=1, a_{4}=2$

| $a_{1} \backslash a_{2}$ | 0 | 1 | 2 |
| :--- | :--- | :--- | :--- |
| 0 | 2 | 1 | 0 |
| 1 | 3 | 0 | 1 |

By a leap of faith, we think $g$ is periodic in $a_{2}$ and $a_{4}$ with period 3 and $g$ is periodic in $a_{1}$ and $a_{3}$ with period 2.
Now, do you see a characterization of the set $A=\{p: g(p)=0\}$ ?
Hmmm..., yes. Denote by $r \in\{0,1\}$ the residue of $a_{1}+a_{3}$ modulo 2 , and by $s \in\{0,1,2\}$ the residue of $a_{2}+a_{4}$ modulo 3 . Then $A=\{p: s-r=0\}$.

## Solution.

Denote by $r \in\{0,1\}$ the residue of $a_{1}+a_{3}$ modulo 2 , and by $s \in\{0,1,2\}$ the residue of $a_{2}+a_{4}$ modulo 3. Let $A$ be the set of positions satisfying $s-r=0$, and let $B$ be the complement of $A$.

Clearly, each of the eight moves
$a_{1} \mapsto a_{1}-1$,
$a_{2} \mapsto a_{2}-2$,
$a_{3} \mapsto a_{3}-3$,
$a_{4} \mapsto a_{4}-4$,
$\left(a_{1}, a_{2}\right) \mapsto\left(a_{1}+2, a_{2}-1\right),\left(a_{1}, a_{2}, a_{3}\right) \mapsto\left(a_{1}+1, a_{2}+1, a_{3}-1\right)$,
$\left(a_{2}, a_{4}\right) \mapsto\left(a_{2}+2, a_{4}-1\right)$, and $\left(a_{1}, a_{3}, a_{4}\right) \mapsto\left(a_{1}+1, a_{3}+1, a_{4}-1\right)$
changes a zero value of $s-r$ to a nonzero value, that is, any move from a position in $A$ must end up with a position in $B$.

On the other hand, there is always a possible move from a position in $B$ back to a position in $A$, as displayed in the table.

| Positions in $B$ | Winning moves to $A$ |
| :--- | :--- |
| $s-r=-1$ | $a_{1} \mapsto a_{1}-1$ if $a_{1} \geq 1$ <br> $\left(a_{1}, a_{2}, a_{3}\right) \mapsto\left(a_{1}+1, a_{2}+1, a_{3}-1\right)$ if $a_{3} \geq 1$ |
| $s-r=1$ | $\left(a_{1}, a_{2}\right) \mapsto\left(a_{1}+2, a_{2}-1\right)$ if $a_{2} \geq 1$ <br> $\left(a_{1}, a_{3}, a_{4}\right) \mapsto\left(a_{1}+1, a_{3}+1, a_{4}-1\right)$ if $a_{4} \geq 1$ |
| $s-r=2$ | $a_{2} \mapsto a_{2}-2$ if $a_{2} \geq 2$ <br> $\left(a_{2}, a_{4}\right) \mapsto\left(a_{2}+2, a_{4}-1\right)$ if $a_{4} \geq 1$ |

Since both the starting position $\left(a_{1}, a_{2}, a_{3}, a_{4}\right)=(1,0,0,4)$ and the ending position $(0,0,0,0)$ are in $A$, the second player can always force a win.

## Comment.

The winning strategy works for any starting position $\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$. The winner, of course, depends on whether this starting position is in $A$ or $B$.

## Further question.

What if we allow sticks of lengths $5,6,7, \ldots$ ?

## 5. Further readings

For more examples to use in teaching, see
[1] D. Fomin, S. Genkin \& I. Itenberg, Mathematical Circles, AMS, 1996
[2] A. Engel, Probelm-Solving Strategies, Springer, 1998
For more serious research interests, see
[3] E. R. Berlekamp, J. H. Conway \& R. K. Guy, Winning Ways for Your Mathematical Plays, Vol. 1-4, 2nd Ed., AK Peters, 2001-2002
[4] J. H. Conway, On Numbers and Games, 2nd Ed., AK Peters, 2001
[5] R. J. Nowakowski (ed.), Games of No Chance, Cambridge, 1996
[6] R. J. Nowakowski (ed.), More Games of No Chance, Cambridge, 2002

