What is Mathematics? Toward A Global View

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With the increase in knowledge in the late twentieth and early twenty-first centuries, the question of what mathematics is has become ever more relevant. It is now clear that what we in the West consider mathematics has a far more diverse history and has been practiced by a far wider group of peoples than was previously thought. As late as the mid-1900's, many mathematics historians thought of mathematics as a primarily Grecian and European creation. Also, non-literate peoples were thought to have a primitive idea of mathematics at best, limited to perhaps simple arithmetic and counting. Recent improvements in scholarship in the history of mathematics, along with the study of mathematics in traditional peoples, now called ethnomathematics, have challenged these long-held views. We discuss these findings, and ask what impact they have on the question not only of where and how mathematical ideas developed, but also on the broader issue of what should be considered mathematics.

We will consider three main questions in this article. First, where does mathematics come from? Second, what does the practice of mathematics look like in non-Western societies? And thirdly, based on the answers to the first two inquiries, we must consider the question of what actually constitutes mathematics, and whether the modern Western view might be too restrictive.

The Traditional View and why it needs to be revised

For most of the modern era, the commonly held view of the origins of mathematics was that no "real" math existed until the Greeks, starting with Thales of Miletus about 600 BC, developed the idea of demonstrative mathematics. This centered on the idea of proving assertions and not just using set rules to do calculations. This idea was passed on to later scholars, notably Euclid, Archimedes, and others. With the fall of the Greco-Roman Empire and the coming of the Dark Ages to Europe, mathematics, this view held, ceased to exist for almost 1000 years.

During this period, there were no advances in mathematical learning, although the Arabs translated and so kept alive the results of the Greeks and Romans. Later, scholars and travelers brought these results to Western Europe, sparking the Renaissance and ushering in the modern period of mathematical and scientific progress.

This view, although widespread, has recently been challenged by the work of many mathematicians, historians and archaeologists. Even in ancient times, Greeks such as Herodotus acknowledged the debt of the Greek mathematicians to the Egyptians, especially in geometry. Also, work on Arab learning during the Middle Ages leads to questions regarding whether it is fair to characterize their contributions as merely translating and "keeping alive" Greek learning so that those in the West could later uncover it. In short, is it fair to say mathematics is a development of only the Greeks and modern Western Europeans? Many would now disagree.

This "Eurocentric" view of mathematics history can in part be explained, perhaps, by the human preference for that with which we're most familiar. Also, historians knew little of the other cultures which have contributed so richly to mathematics before sometime in the early to mid twentieth century. In some cases, such as with the Chinese and Indians, languages were not well known among scholars. In others, such as with the ancient Babylonians, records were few and not yet deciphered. However, even with these caveats, enough information existed that called into question the traditional view of mathematics history that it's surprising that so many have held to it with such tenacity.

For example, consider the origin of the "Western" system of numerals. The Indians originally developed a base ten positional system which was then passed on to the Arabs. From there, Fibonacci's *Liber Abaci* was the most important work in introducing this decimal place value system to the West. In contrast, Greek and Roman numerals weren't place value at all, so the idea of a place value system can't be considered to be either Grecian or European in origin. In fact, the only cultures which independently developed such a system were India, the Maya, Babylonia, and China. Also, once the Indoarabic numerals were introduced in Europe, those who wished to continue the use of Roman numerals often fought against the newer system. Of course, the number system we now use eventually won out. Some intellectual historians now view this change as even more important than it might at first seem, even going so far as to suggest that only after the Indoarabic numerals were widely accepted did Western Europe begin to experience a true Renaissance in mathematics.

Another very important idea in the history of mathematics is that of zero. Again, this concept did not come to the West via the Greeks, but rather from India. The concept of a symbol for zero used much as we do was even rarer than that of place value notation, originally occurring only among the Mayans and Indians.

Finally, though it is hard to argue that the Greeks pushed the study of geometry to heights never before seen, they acknowledged a debt to the Egyptians. All cultures, even very primitive ones, have geometric concepts, and some of them are remarkably sophisticated. The Greeks formalized these concepts, adding the idea of proof. For this, it's clear all of modern mathematics owes them a great debt. The question is, in recognizing this are we ignoring equally remarkable contributions from other cultures?

In answering this question, we should perhaps consider the fact that the idea of "proof" became much more rigorous in the nineteenth and early twentieth centuries. In fact, it's not at all clear that many earlier proofs, even by figures considered to be among the elite of mathematics, would hold up to modern scrutiny. So is the modern emphasis on rigor, especially in analyzing the history of mathematics and the exploration of mathematics in other cultures, resulting in too narrow a perspective on mathematical thought and its origins?

The following two quotes, by notable historians of mathematics, are quite revealing of the biases some have held.

“The History of mathematics cannot with any certainty be traced back to any school or period before that of the Ionian Greeks." (Rouse Bell, 1908, p. 1)

“[Mathematics] finally secured a new grip on life in the highly congenial soil of Greece and waxed strongly for a short period… With the decline of Greek civilization the plant remained dormant for a thousand years…when the plant was transported to Europe proper and once more imbedded in fertile soil.” (Kline, 1953, pp. 9-10)

It is notable that the first quote comes from a historian of the early twentieth century, and at the time many of the findings addressed in this paper weren't known.

However, by the time of the second quote, in Morris Kline's work on *Mathematics in Western Culture*, this excuse can't be made for his bias against the mathematical works abundant in other cultures. In particular, the mathematical works of Egypt, Babylonia and the Arabs were well known at that point, and Kline's failure to recognize them raises questions about the reasons behind it.

Many cultures, for example, did not have people whom we would consider mathematicians at all, primarily because the ability to have scholars who devote their lives to certain specialties implies a fair amount of wealth, so that individuals can be free from more practical pursuits. The ancient Greeks were fortunate enough to have this luxury; many other cultures did and do not, though this doesn't necessarily mean they did not have mathematical ideas worthy of note. In particular, if one considers the scribes, engineers, merchants and priests in ancient cultures, one will find quite a bit of advanced mathematical thinking. These considerations lead us to look again at the history of mathematics in a broader light, considering cultures that have been previously neglected.

Mathematics History – New Perspectives

If we consider the history of mathematics from a broader perspective, it is clear that math came from many distinct sources throughout the world. For instance, the Egyptians and Babylonians contributed a great deal to the "Greek miracle." Also, the Indians and Chinese had very advanced systems, many of whose ideas passed to the West. They also developed many mathematical concepts which are credited to Western sources, but predate those in the West. Finally, a reexamination of the Arab contribution to mathematics in the Middle Ages is clearly necessary for a full understanding of the origin of many mathematical concepts.

This paper cannot, of course, discuss all of these cultures in detail. We will limit ourselves to a recounting of some notable achievements of three of them, the Babylonians, the Chinese and the Arabs. The interested reader is encouraged to seek out more information on these and the other cultures important in mathematics history. George Gheverghese Joseph's book *The Crest of the Peacock* contains a brilliant discussion of much of this information. In addition, there are more and more online resources available on mathematics history, most notably *The MacTutor History of Mathematics archive*, at http://www-gap.dcs.st-and.ac.uk/~history/.

One culture which had very advanced mathematical ideas, some dating back to almost the third millennia BC, is the Babylonians. They had a base 60 positional number system, and have passed this on to us in our counting of minutes, seconds, and degrees in geometric figures. To save their scribes time, they possessed tables for many common calculations, including computing reciprocals, squares, cubes, and exponential functions. The latter were probably used for computations with interest, some of which are mentioned in other tablets that have been found.

Some of their geometrical works are particularly impressive, and it is still unclear how these results were obtained. Two outstanding examples of their geometry are the Plimpton 322 and Yale tablets.

A photograph of the Plimpton 322 tablet is shown below (Figure 1). It is a table consisting of five columns of numbers dating from the Old Babylonian period, between 1600 ad 1850 BC. Reproducing part of this in modern notation, we have the five columns of numbers shown below (Figure 2). Because the tablet is partially broken, it is unclear what all the entries are. However, if we interpret the second row, labeled b, as the base of a triangle, the fifth, labeled h, as the height, and the third, labeled d as the length of the hypotenuse, we see that the numbers in these rows are Pythagorean triples.

?	b	d	Row #	h
	119	169	1	120
	3367	4825	2	3456
	4601	6649	3	4800

Table 1 A part of the Plimpton 322 Tablet reproduced in modern notation.

Figure 1 Photograph of the Plimpton Tablet. Photo from O'Connor and Robertson, 2000.

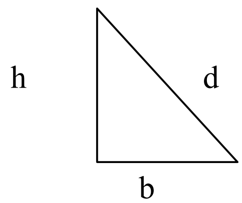

Figure 2: A triangle labeled to correspond to the Plimpton 322 Tablet as reproduced in Table 1.

Although all the numbers in the table clearly fit this explanation, how the Babylonians got such large triples and why they would have recorded them is another matter entirely. One reasonable suggestion seems to be that they might have independently derived Diaphanous' method for calculating whole number Pythagorean triples. That is, given two integers *m* and *n,* a Pythagorean triple is given by $2mn$, $m^2 + n^2$ and $m^2 - n^2$.

Another amazing tablet produced by the Babylonians is the Yale tablet. This tablet consists of a drawing of a square with its diagonals also included. Next to one side of the square is the Babylonian representation of our number 30. Near the intersection of the diagonals are the numbers we would represent as 1.41421297 and below that 42.4263891. No explanation is given. However, based on the geometry and how close the number 1.41421297 is to $\sqrt{2}$ (1.41421356 to the same number of places), it is clear that the tablet shows how to calculate the diagonal of the square by giving a value for $\sqrt{2}$ and $30\sqrt{2}$, which would be the length of a diagonal of a 30 by 30 square.

This tablet, like the Plimpton one, dates from the Old Babylonian period. In addition to showing that the Babylonians knew of the Pythagorean theorem, over 1000 years before Pythagoras lived, it also shows an extremely accurate approximation of $\sqrt{2}$. It is not at all clear how they derived this estimate, however. I have come across two plausible suggestions. The first is that they had somehow developed Heron's method, obviously well before Heron lived. The second is that they may have used bisection, since they had a number system that allowed for computational ease and their other work suggests the willingness to do involved computations. Whatever is the case, it is amazing that an estimate of an irrational number to such a high degree of accuracy existed so early in recorded history.

Finally, although we have focused on the geometrical work of the Babylonians, they had a very sophisticated understanding of the solution of algebraic type problems. Early algebra developed in three phases. First there was rhetorical algebra, characterized by the telling of story problems with an algorithm given for solution. Next, syncopated algebra developed. In this stage, phrases or terms were used to stand in for what would later be represented as symbols such as *x* and *y*. Finally, symbolic algebra developed, which is the way in which we represent and solve equations now. The Babylonians did not use symbols, but neither did any culture until the Arabs and Europeans in the second millennia AD. Although algebraic problems were stated in the context of story problems, such as in the rhetorical algebraic traditions, the Babylonians also used phrases, such as length, breadth, square, area and volume to stand in for what we would now denote symbolically as x, y, x^2, xy, and xyz.

Yet another culture whose work has far too often been overlooked is the Chinese. The Chinese were one of only four cultures, as previously mentioned, with a place value system. Their rod numerals were a decimal place value system that allowed the easy representation of any positive integer. In addition, they had conventions whereby they could represent decimal fractions and even negative numbers. We find references to all of these concepts in works dating back to the period around the birth of Christ.

The Chinese excelled in both algebraic and geometric work. A number of Chinese mathematicians obtained increasingly accurate approximations of π using inscribed polygons This culminated with the result of Tsu Chung Chih in c. 480 AD, which used polygons with up to 23576 sides and the Pythagorean theorem to compute successively better approximations to π. His result was accurate to six decimal places.

They also had developed Pascal's triangle of binomial coefficients by about 1000AD. Chia Hsien, the mathematician responsible, recorded the triangle up to the sixth degree coefficients. It was used in solving equations. This development predated Pascal's by over 500 years. The Chinese were hardly the only culture outside of Europe to know of and use this triangle. There are also references to it in India, where it was used in combinatorics, and, as we will later see, among the Arabs.

Perhaps the most remarkable work the Chinese produced in mathematics is the *Nine Chapters of the Mathematical Art*, believed to date from the beginning of the Christian era. This book is the first applied math text in history. If to be considered educated in the West required knowledge of Euclid's *Elements*, the *Nine Chapters* was equally important in the East. It formed the basis of mathematical training for officials, not only in China but also in Korea and Japan. Some topics that were covered included proportions, the rule of false position, finding square and cube roots, and Cramer's rule for solving systems of two equations in two unknowns. There was even a treatment of matrix methods used to solve larger systems of equatoins. In geometry, complex areas and volumes were treated for engineering purposes. The last chapter considered problems using the Pythagorean theorem.

Later Chinese mathematicians wrote commentaries on this work, explaining and extending the results in it. Among these were Liu Hui around 200 AD and Yang Hui in about 1200 AD. Unlike Euclid's work, which has been translated numerous times into English and other languages, there is as yet no complete English translation of the *Nine Chapters*. Given its importance in Chinese intellectual history, this omission is amazing.

Finally in reexamining the history of mathematics we cannot neglect to mention briefly the amazing work of the Arabs. To understand Arab contributions, we must know something about the history and culture of the Islamic world.

In 622, Muhammed went from Mecca to Medina, forced to flee because of his preaching of a new monotheistic religion. In only ten years, in 632, he died. However, by then he had founded a movement which within only one hundred years claimed a vast empire. At its height, the Islamic movement controlled the region from Northern

Africa to the borders of China and into Northern India, and included Spain in Europe up to the border of France.

The two main dynasties ruling this civilization were the Abbasids, centered in Iraq, and the Umayyads, centered in Spain. In 762 the Abbasid caliph moved his capital to Baghdad; his goal was to build a new Alexandria. The next two caliphs, al-Rashid and al-Mamun, continued this construction, building an observatory, a library and an institute for translation and research called the House of Wisdom. This was to be the focal point for Arab learning for the next two hundred years. Numerous scientists and translators were affiliated with the House of Wisdom. The caliphs often provided patronage. By "Arab" scholars, it is important to recognize that we do not just mean Muslims or those who were of Arab ethnicity, though many did fit this description. Any one who was working within the Islamic world could be included, which meant, for example, that looking at two figures we will discuss later, Omar Khayyam was Persian, and Thabit ibn Qurra belonged to a religious sect called the Sabeans, deriving from the Babylonian star worshippers.

We have already touched on the role of the Arabs in the spread and development of Indian numerals. But they accomplished far more than that from a mathematical and scientific perspective. The scholars at the House of Wisdom translated many Classical texts. Early on they translated the *Elements.* Afterward mathematicians stated and proved theorems rigorously in imitation of this text. We consider just a few of the notable Arab mathematicians and their achievements below.

Perhaps the most famous and influential of the Arab mathematicians was Al-Kwarizmi. He lived from about 780 to 850, and came to Baghdad at the behest of Caliph al-Mamun in about 820. After serving as chief astronomer, the caliph made him the head of the House of Wisdom. He produced many scientific texts, but the two most important were *Algorithmi de Numero Indorum* (the Arithmetic) and *Hisab al jabr w'al-muqabala* (the *Algebra).* These two books had a profound effect on the history of mathematics. The *Arithmetic* described the new Indian decimal numeral system and how to compute with it. It was translated into Latin three hundred years later and proved an important tool for Europeans in learning the new number system. It also is the origin of the term "algorithm," from the Latin “digit Algorismi” or, "Thus says Al-Kwarizmi".

Al-Kwarizmi is also fundamental in the founding of algebra, and because of his work it became an important part of Arabic research which was later passed to the Europeans. The *Hisab al jabr w'al-muqabala* (or in English "restoration and compensation") is where we get the term *algebra*. It is an algebra text which treats quadratic equations, geometry, linear equations, and the application of mathematics to inheritance problems. Al-Kwarizmi's work in algebra did not involve symbols but rather would be classified as syncopated algebra, with words standing in for variables. It was not until the fourteenth century that Arabic mathematicians started to use symbols and by the fifteenth century they had developed a truly symbolic algebra.

Another notable mathematician was Thabit ibn Qurra, mentioned above. He was born about 836 in northern Mesopotamia and died in 901. He became one of the most important translators in Baghdad, and he established a school for translators in the Arab capital. He translated the *Elements*, works by Archimedes and several other important Greek works. These were translated into Latin in the twelfth century by the Europeans, and hence helped spur the rebirth of European mathematics. He was more than a translator however. His original work included a rule for discovering pairs of amicable numbers, an attempt to prove the parallel postulate, and a proof of the Pythagorean theorem which resembled an earlier proof found among the Chinese

The final mathematician from the Arabic world we will mention is Omar Khayyam, perhaps best known as the poet who wrote the *Rubaiyat.* However, Khayyam, who was born about 1040, also was a gifted mathematician, particularly in the area of algebra. In his major work on algebra he classified equations by degree, gave rules for solving quadratics, and used geometry to solve cubic equations. He also, like the Chinese mathematicians earlier mentioned, knew of and discussed Pascal's triangle. Finally, his work on proportions served to extend Euclid's concept, which applied only to rational quantities, to positive irrational numbers such as $\sqrt{2}$. In so doing, he was one of the first to consider a more rigorous treatment of real numbers, which was only taken up by European mathematicians much later.

From these few overviews, we may see that the Arabs played a much more important role in the history of mathematics then as translators and custodians of knowledge. Given the current world situation, most notably the recent war in Iraq and the terrorist attacks of 9/11, perhaps the best way to conclude this summary of Arabic contributions to mathematics is a quote from the online MAA publication*, Devlin's Angle*, written by Keith Devlin. Quoting form the July / August, 2002 article, "The Mathematical Legacy of Islam. " "... [T]he culture that these fanatics [Arab terrorists] claim to represent when they set about trying to destroy the modern world of science and technology was in fact the cradle in which that tradition was nurtured. As mathematicians, we are all children of Islam." Whether one agrees fully with Devlin's analysis is open to debate, but it is clear that the Arab culture contributed significantly to the development of mathematics and science, and that without this contribution the modern world, and certainly modern mathematics, would look remarkably different.

Mathematics in Traditional Cultures: An Introduction to Ethnomathematics

In addition to a reexamination of the history of mathematics, scholars are also considering new information regarding mathematics in traditional cultures. Most cultures are non literate, making the study of their practices and knowledge dependent on field work and oral traditions as opposed to written documents. Many traditional cultures consist of small groups which are relatively isolated, and are difficult to study due to complications arising from lack of knowledge of the language as well as other factors. These cultures may have extremely sophisticated ideas, but it is only recently that anthropologists, ethnologists and mathematicians have started to analyze them. This new field is called *ethnomathematics.*

One area that has been of special interest to ethnomathematics is graph theory. The traditional view of this discipline is that Euler originated it in the eighteenth century while answering a question about the bridges of the town of Konigsberg. Could a person travel once over all seven bridges which went between the four landmasses in the town without retracing any path? In answering this question, Euler defined the notion of a *graph*, a set of points called *vertices* some of which are connected to each other by curves called *edges*. Any graph which has a path such as the one described above is said to have an *Euler path*; if the path ends and starts at the same vertex it has an *Euler circuit*.

Other groups clearly may have situations leading to much the same questions Euler answered. So it is perhaps not surprising that other groups have come up with diagrams and procedures often nearly identical to those which Western mathematicians would classify as graph theory. One example of such a group is the Bushoong, who live in Africa. Among these people there is a children's game where the goal is to trace a figure continuously without duplicating any line, and to start and end at the same point. To us this task would be identified as finding an Euler circuit on a graph. The anthropologist whom the children first challenged was unable to succeed in tracing the figure correctly. In relating the story he did not realize that there was an advanced mathematical subject studied by professionals which addressed the same issues.

There are other cultures in which the ideas of graph theory play an even more important role. The Tshokwe are one such group. They live in northeast Angola and Zaire in small villages under family chiefs. The tracing of diagrams called *sona* is part of a storytelling tradition carried on by the men. Western mathematicians examining *sona* have found that they are in fact graphs.

Among the Tshokwe these *sona* convey important cultural traditions. For example, when a young man is to pass into adulthood he and a group of other boys who are also ready go through a series of rites called the *mukanda*. After a ceremony marking their approaching manhood, they live in a separate camp visited only by their fathers and certain other males for one to three years. At the end of this time, they return to the village, receive new names and clothes and return home as adults. There are several *sona* dealing with the *mukanda*, most notably one which shows how the young men and their guards, represented by dots within the graph, are inside the camp, whereas others (still more dots) are outside and can't enter.

Another similar *sona* tells a tale of thieves trying to steal a dead chief's bracelet, which is his symbol of authority and so should only be passed on to his successors. The thieves are again represented by dots surrounded by the rather complex graph which signifies the village. Again, the emphasis is on the distinction between those who are on the inside and outside of the simple closed curve which represents the village where the chief is. In these two *sonas*, we can see the Western mathematical concept of the Jordan Curve theorem, which states that a simple closed curve in the plane divides it into two regions, the inside and the outside.

Yet another Tshokwe story illustrating the use of *sona* to emphasize the division of the plane into regions involves a man with a gossipy wife. He builds barriers between her and the neighbors so that she can't gossip with them. What is especially interesting about this story is that there are two distinct *sonas* which can be drawn to illustrate it, and in graph theoretic terms, although they appear different, they are isomorphic. That is, their vertices and edges may be placed in a one-to-one correspondence to each other.

A final point about Tshokwe *sona* is that most of them are regular graphs of degree four. For a graph to be regular means that all of the vertices have the same number of edges meeting at them, or in graph-theoretic terms the same degree. So a regular graph of degree four has four edges meeting at every one of its vertices. The figure below shows an example of such a graph.

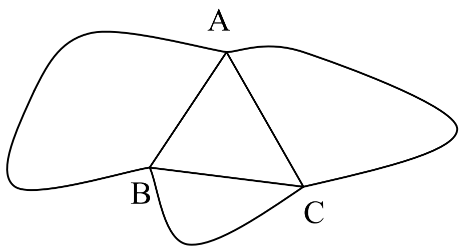

Figure 3: A regular graph of degree four. That is, all the vertices have four edges meeting at them.

Yet another traditional people for whom graphs hold an important cultural place are the Malekula who live in the Republic of Vanuatu, formerly known as New Hebrides, in the South Pacific. Among the Malekula the men pass down the knowledge of how to trace *nitus* from generation to generation. Again, we would call these *nitus* graphs. Also, as in the Western study of Euler circuits and paths, it is important that the tracing of the *nitus* be done without going back over any path again, and that all edges be covered. Further, it is considered so important whether one can start and end these tracings at the same point without any backtracking that those that have this property are given the name of *suom*. Of course, Western mathematicians would say that graphs classed as *suom* have Euler circuits.

The tracing of *nitus* is of such consequence among the Malekula that it is used in relating important myths. One example concerns the origin of death, believed to be the result of the killing of Barkulkul by his brother Marelul. This story is conveyed by means of a *nitu*. Further, knowledge of how to trace *nitus* is considered important both in this life and for what comes after. In order to enter the Land of the Dead, the spirit of the Malekula is challenged to complete the tracing of a *nitu*. Failure results in being eaten by an ogre!

Finally, unlike with the graphs of the Bushoong and Tshokwe, in the case of the Malekula we know the exact paths used to trace over one hundred figures. They were recorded by an ethnologist who felt that this tradition was unique and warranted further study. Because of this, mathematicians have been able to study the *nitus* and the paths used to trace them in detail. From this analysis we know that the *nitus* range from simple closed curves to graphs having more than one hundred vertices, some of degree up to twelve. Also, the Malekula were clearly concerned with tracing any figures which allowed for Eulerian paths and circuits in this way if at all possible, and for the most part, they succeeded in doing so.. Also, Marcia Ascher, a noted ethnomathematician, analyzed the tracing procedures that the Malekula used and found that they could be explained by what she terms a *process algebra*. As she says, "The word algebra is used in its most fundamental sense: there are variable entities that are operated upon in accordance with specific rules. Here the variable entities are tracing procedures and the rules include processes that transform the procedures into other procedure.".{Ascher, *Ethnomathematics*, p. 51). She also found that only certain transformations were allowed, and she used this algebraic system to actually reproduce and trace many of the Malekula figures in the traditional way.

Yet another area in which we see mathematical concepts emerge from traditional cultures is in the playing and design of games of strategy and chance, and the solving of puzzles. Many of these are far more important in the life of the culture than similar recreations are in the Western world.

One example of such a game is Dish. This is a game played by Native Americans in many parts of North America, including the Cayuga, the Seneca, and the Cherokee.

There are some variants in the way Dish is played, so we will consider the rules and context of the game among the Cayuga. This group used a set of six peach pits blackened on one side and a wooden bowl. The pits were tossed, and if they landed all six on the same side (either burned or not) , the player scored five points; for five on the same side, the player scored one point. No points were earned for any other combinations. Also, if the player earned points he received an extra turn. Otherwise he had to pass the bowl to the opponent. The game was played until one player reached some total number of points such as 100.

Interestingly, if we look at the probabilities of the outcomes that occur due to the tosses in Dish, we note that the least likely outcomes (all on the same side) receive the most points, and the most likely the least points. The probabilities associated with each outcome by the binomial theorem are 1/64 for all burnt or none burnt, 6/64 for five on one side, and 15/64 or 20/64 for four or three of a kind respectively. As Ascher points out in *Ethnomathemaitcs*, if we assume that five points is given to the least likely combination, the number of points assigned by the Cayuga closely corresponds to the number that would be assigned by probability if they had known this mathematical theory. The chart below illustrates this. If we start with a point value of five for all burnt sides up, then all burnt sides down should be assigned the same point value since this outcome is equally likely, and, in fact, we find that these two outcomes are both assigned five points in the game. According to the binomial theory, the next outcome, five up of one type (or one up of the other type), is 1/6 as likely to occur. Its point value should therefore be $\frac{1}{6}\times 5$, or $\frac{5}{6}$. In fact the point value for this outcome in the game is the closest integer to this, or 1. We can compute the point values for the other outcomes similarly. If we do this, we get 1/3 points for four of a kind and $1/4$ points for three of a kind. We note that among the Cayuga these outcomes are given the closest integer value to what we have computed which is 0 points.

No of same sides (burnt or not)	6	5	4	3	2	1	0
Probability of occurring	1/64	6/64	15/64	20/64	15/64	6/64	1/64
Point value from probability	5	5/6	1/3	1/ 4	1/3	5/6	5
Point value assigned	5	1	0	0	0	1	5

Table 2 A probabilistic analysis of the outcomes and point assignments in Dish.

Clearly, the Cayuga did not know the mathematical theory of probability, but the correspondence between the numbers above shows that they had a very good intuitive notion of the likelihood of the outcomes and how each should be weighted in the scoring system of the game.

Finally, to have a fuller understanding of Dish we must explain that this game had much more importance among the Native American cultures then we would assign to similar games now. As one example of this, among the Iroquois a sick person might ask for a game of Dish to be played in order to secure healing. The whole community would respond, with the village of the sick person inviting another village to join in. The games could last up to six days and sometimes went on all night. People would bet on the outcome using goods collected because of messages received in dreams. Because of this, the whole life of the village could be disrupted and people might lose large amounts of property. However, the belief that it was important to show support for the ill person was paramount. In fact, the Iroquois viewed some dreams as symbolic of a person's unspoken desires and felt that if these desires were not expressed and attended to, illness could result. This understanding of the psychological underpinnings of health anticipated theories of the mind advanced by Freud and many others by a number of centuries.

A second example of the importance of games in other cultures is Mancala. There has been much written on this subject and its analysis by mathematics. We point out, that in addition to having complex strategic underpinnings, the game was far more than a recreation in most African cultures. In fact, adeptness at Mancala demonstrated the wisdom of the chief and in some cultures a competition was even used to pick a chief.

The final game we will examine is a logical puzzle known as the river-crossing puzzle. Virtually identical versions are found among groups in many parts of Africa, even though these groups are isolated from each other. It also occurs in Europe, where the first written form is usually attributed to Alcuin of York, who was a theologian from the late eighth century. However, it is also found in the folklore of the Welsh, Saxons, Russians, Italians, and others. It has clearly circulated in European cultures for well over a thousand years. We have no way of knowing how long it has been in the cultural tradition of the African groups.

In almost all instances the structure of the puzzle is the same. There are three objects, for example a wolf, a goat and a cabbage. A man must get these safely across a river, but he has a boat which will hold only himself and one of them. Also, clearly the wolf can't be left alone with the goat or it will be eaten, and the goat can't be left alone with the cabbage or it will be eaten. There are two solutions. Either first take across the gaot and leave it, then come back for the cabbage, put it on the other shore and bring the goat back. Take across the wolf and leave it, which is safe since it and the cabbage are now on the far side and can be left alone together. Finally, go back and bring the goat across. All three are now on the same side of the river, and the puzzle is solved. The second solution switches the wolf and the cabbage in the above narrative. There is also a slightly different version of the puzzle in parts of Africa where the person can take two of the three items with him, but does not have enough control so that they won't attack each other in the process. Therefore he can't take the goat and the wolf on the same trip, for example. Variants on a solution similar to that mentioned above for the first description of the puzzle also exist for this version.

The context in which the story is told in African cultures makes it clear that shrewdness and logical reasoning are highly prized. Consider the version told among the Kpelle of Liberia. A young man wishes to marry the king's daughter. She agrees, but the king sets him a puzzle which he must solve. If he fails, the two can still marry but he will have to pay a bride price to the king. The king has a cheetah which has been trained to eat any fowl it sees. He also has several fowls and some rice. The puzzle is to get the cheetah, a fowl and the rice across the river in a boat which will only hold the suitor and two of the three items. Also, he can't control the items while on board the boat, so for example he can't take across the cheetah and the fowl on the same trip. The young man tries to solve the puzzle and at first cannot. He asks his father for help. Although the father gives him replacements for the rice and fowls he has lost, he warns his son that he must solve the puzzle himself or else he will be shamed and start to think ill of himself. When the young man finally solves the puzzle, the families have a joyful celebration and the couple is married. The point is that the Kpelle feel that individual achievement reflecting individual effort and cleverness is important, and this is reflected in their telling of the story.

A final example of the puzzle, distinct from all others found elsewhere in Africa and Europe, occurs among the Ila of Zambia. Instead of three items to be transported there are four, a leopard, a goat, a rat and a basket of kafir corn. The boat again can only hold the man and one of these items. It can be shown that with this change the problem can't be solved logically while keeping all four of the items intact. Westerners would probably resolve the dilemma posed by sacrificing the rat. However, this is unacceptable to the Ila because of their ethical beliefs. First, they believe that one may temporarily assume the form of or pass into a plant or animal either briefly during life or at death Also, a traveler is held responsible for his companions on a journey. To allow anything to happen to someone in this situation is a serious violation of Ila law. So what is the solution to the puzzle among the Ila? There is no right solution other than to stay where the person is, and not cross the river at all!

The final example of mathematical ideas in traditional cultures that we will consider is navigation among the Caroline Islanders. The Caroline Islands is an archipelago extending for 1500 miles from the east to the west, that is north of New Guinea. Many of the islands are extremely small. The islanders survive mainly by fishing, both in the deep sea and among the reefs and lagoons. Due to the fact that most of the islands are only a foot or two above sea level, and the populations are very small because of the islands' size, storms and illness can cause great hardship. Canoes and sailboats are the major form of transportation. Communication between the islands is necessary to help in emergencies as well as for social interactions and trading.

Sailing between the islands usually requires going out of sight of land for long periods. So a special group, the navigators, has arisen to undertake this dangerous task. They have no navigational equipment, just their boats and mental spatial models. These enable them to conduct numerous voyages among the islands, both of the archipelagos and even to those further away. For example, in 1962- 63 there were fifty-seven voyages undertaken between the three relatively close islands of Lamotrek, Elato, and Satawal. By "close," however, we mean that Elato is fifteen miles from Lamotrek and Satwal is more than twice that distance. Also, the navigators have sailed as far afield as the Marshall Islands, New Guinea and the Philippines, and they used to make an annual trip to Guam. Just recently a navigator made a journey to Siapan in the Mariannas. During the course of this voyage, he was out of sight of land for 450 miles.

The navigators are an insular group considered to hold the most prestigious occupation among the Islanders. Knowledge is shared freely within the group, and new navigators come from close kin or those who are willing to pay large amounts of money for the navigators' secrets. The navigator class has certain customs which aid in keeping their information secret; for example, they can only eat with each other, and all their food must be prepared and served separately.

Looking at the modeling they use to sail successfully between the Pacific Islands, we find that there are two separate spatial models employed. One involves the learning of a star compass, where the rising and setting of the stars with reference to the circle of the heavens must be memorized. In addition, the navigator must know the relative locations of the islands with respect to the stars and the directions to sail from one to another. The other model involves the use of a reference island for each voyage that is out of sight of both the starting point and the destination. The boat is viewed as stationary, with everything else moving around it as a fixed unit. The journey is accomplished in several stages, with the fixed boat being passed by the reference island under a sequence of stars which the navigator must know. The idea of the boat being fixed and everything else moving reminds the author of modern relativity theory, although there are differences. What is clear is that even though this model is not truly accurate, it allows the navigator to successfully undertake and complete long trips.

Conclusion – New Questions and Considerations

In closing, we pose the same question asked at the beginning. What is mathematics? Do we, in the modern Western world have too narrow a view of this subject? It is clear that at least the history of mathematics is far more global than has been previously thought. Many Western mathematical concepts either originated in or were independently discovered by cultures outside of the Greco-European sphere of culture, such as the Babylonians, Chinese and Arabs. Now that we are learning about these other mathematical sources, it may change our view of the discipline of mathematics as a whole. Further, in examining the mathematical ideas of traditional cultures, we see that even societies we would have considered "primitive" in the past, with no writing or great cities, can have very advanced notions involving space, number, time, and pattern. Instead of being a narrowly defined specialty practiced by a small group as it is in our culture, perhaps mathematics is a far more intrinsic part of all cultures then previously realized. In many cultures, certain mathematical ideas are very important in defining core beliefs, such as the Malekula's belief about the Land of the Dead and the tracing of *nitus*. Would it be more productive (and interesting!) to view mathematics not as a narrowly defined Western specialty, but more as a set of ideas which can form a defining part of culture, much as language, kinship and spirituality do? Perhaps this is so, if we broaden what we mean by "math."

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