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In the beginning, there was Euler...

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0. Introduction

George Pólya had three commandments for math teachers, the last of which is that they should have a healthy regard towards problem solving. He himself followed this commandment and made numerous contributions to the Problem Section of *The American Mathematical Monthly*.

My own limited contact with problem sections of math journals convinced me the value of problem solving in teacher's professional development and in enriching the classrooms. In the introduction to the first issue of the *Monthly*, the founding editor Benjamin F. Finkel said [**12**]: "The solution of problems is one of the lowest forms of mathematical research, ... yet its educational value cannot be overestimated. It is the ladder by which the mind ascends into higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem."

Some of these problems often have interesting historical connections, which help to motivate the more polished and linear presentations of texts and lectures. In this talk, we present several examples, all of which trace back to Euler, one of the best problem solvers of all times.

1. How fortunate are the Metropolitans

In 1735, Euler discovered the infinite product $\frac{\sin x}{x} = \prod_{n=1}^{\infty}[1-(\frac{x}{n\pi})^2]$ and consequently revealed that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ [**5**]. This celebrated Euler sum had perplexed many great mathematicians in the previous century [**3**]. But since Euler derived many such results using explicitly the arithmetic of infinite and infinitesimal quantities, he is often portrayed in popular accounts as a reckless symbol-manipulator, who worked in a number system fraught with nonsense and contradiction. Mckinzie and Tuckey [**10**] recently made an interesting effort to rehabilitate Euler in the contexts of hyperreal numbers and nonstandard analysis.

AMM 10866 [2001, 371]. *Proposed by Jerry Kazdan and Herbert Wilf, University of Pennsylvania, PA.* The city of Metropolis is fortunate to have infinitely many bus companies serving its citizens. At a certain bus stop, a bus from company i arrives every i^2 minutes, for every $i \in \{1,2,3,...\}$. A traveler arrives at the bus stop at a random time, with no information about when any previous buses arrived. Find the expectation and distribution of the number of minutes that the traveler waits for a bus.

Solution. Let t be the number of minutes that the traveler waits for a bus, and $F(t)$ the cumulative distribution function of t. Then $F(t) = 1-\prod_{i=1}^{\infty}(1-\frac{t}{i^2}) = 1-\frac{\sin\pi\sqrt{t}}{\pi\sqrt{t}}$ for $t \in [0,1]$ ($\frac{\sin 0}{0}$ is understood to be 1). Hence the density function of t is $f(t) = F'(t) =$

$\frac{1}{2t}(\frac{\sin\pi\sqrt{t}}{\pi\sqrt{t}} - \cos\pi\sqrt{t})$. We then obtain the expectation $E = \int_0^1 tf(t)dt =$ $\frac{1}{2}\int_0^1(\frac{\sin\pi\sqrt{t}}{\pi\sqrt{t}} - \cos\pi\sqrt{t})dt = \frac{1}{\pi^2}\int_0^\pi(\sin u - u\cos u)du = \frac{4}{\pi^2}$.

2. It's difficult enough to be perfect alone

Here, we probably should say: "In the beginning, there was Euclid...". Indeed, Euclid first established the theorem that if $2^p - 1$ is prime then $2^{p-1}(2^p - 1)$ is perfect. This theorem replaced one question: "finding even perfect numbers", with another: "finding primes of the form $2^p - 1$" (Mersenne primes). The most recent discovery is the 39th Mersenne prime $2^{13466917} - 1$ [**8**]. However, it was Euler who proved that (1) an even perfect number is necessarily of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime [**6**]. Euler also turned his attention to the existence of odd perfect numbers and observed: "Whether ... there are any odd perfect numbers, is a most difficult question." But he did establish a nice fact: (2) an odd perfect number must be of the form $p^{4j+1}m^2$ where $p = 4k + 1$ is a prime not dividing m [**6**]. The usefulness of these results is illustrated in the following example.

AMM 10869 [2001, 372]. *Proposed by Lenny Jones, Shippensburg University, PA.* Find every positive integer n such that both $n - 1$ and $n(n + 1)/2$ are perfect numbers.

Solution. We show that $n = 7$ is the only solution.

Case I. If $n = 4k$ then $n - 1$ cannot be perfect by (2).

Case II. If $n = 4k + 1$, then $n - 1 = 4k = 2^{p-1}(2^p - 1)$ and $n(n + 1)/2 =$ $(4k + 1)(2k + 1)$. Since $(n, n + 1) = 1$, one of $4k + 1$ and $2k + 1$ must be a perfect square by (2). If $4k + 1 = l^2$ then $(l - 1)(l + 1) = 2^{p-1}(2^p - 1)$, which is a contradiction since $2^p - 1$ is too big to be a prime factor of $l + 1$ or $l - 1$. We will likewise reach a contradiction if $2k + 1$ is a perfect square.

Case III. If $n = 4k + 2$, then $n + 1 = 4k + 3$. So there must be a prime $q \equiv 3 \pmod 4$ which has an odd exponent in the prime factorization of $n + 1$. The same holds true for $n(n + 1)/2$, since $(n, n + 1) = 1$. Hence $n(n + 1)/2$ cannot be perfect by (2).

Case IV. If $n = 4k + 3$ then $n - 1 = 2(2k + 1)$. Thus $k = 1$ and $n = 7$ by (1).

3. Platonic about polyhedra

Euler characteristic was first observed by Decartes in 1640, but without proof. It was rediscovered and proved by Euler in 1752. Among the many consequences of this formula is a neat way to show that there are only five regular (Platonic) solids [**2**]. A recent *Monthly* problem provides another similar application.

AMM 10856 [2001, 172]. *Proposed by Andrei Jorza, Romania.* Find all bounded convex polyhedra such that no three faces have the same number of edges.

Solution. Let $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ be the sets of vertices, edges, and faces of a desired polyhedron with $|\mathcal{V}| = V$, $|\mathcal{E}| = E$, and $|\mathcal{F}| = F$. Denote by e_v the number of edges meeting at the vertex v and by e_f the number of edges enclosing the face f. Then $F - 2 = E - V$ and $2E = \sum_{v\in\mathcal{V}} e_v \geq 3V$. Suppose F is even. Since no three faces have the same number of edges, $3F - 6 = 3E - 3V \geq E = \frac{1}{2}\sum_{f\in\mathcal{F}} e_f \geq 3 + 4 + 5 + ... + \frac{F+4}{2} = \frac{(F+10)F}{8}$, i.e. $0 \geq$ $F^2 - 14F + 48 = (F - 6)(F - 8)$. Hence $F = 6$ or 8. Likewise, if F is odd then $3F -$ $6 \geq (3 + 4 + 5 + ... + \frac{F+3}{2}) + \frac{F+5}{4} = \frac{F^2+10F+1}{8}$, i.e. $0 \geq F^2 - 14F + 49 = (F - 7)^2$. Hence $F = 7$ in this case. Notice also that $F = 6$, 7, and 8 in fact force all the corresponding equalities to hold. Therefore $(V, E, F) = (8, 12, 6)$, $(10, 15, 7)$, or $(12, 18, 8)$.

PoW [**4**]. Find all bounded convex polyhedra such that no two faces have the same number of edges.
Answer. None. Argue as above, or with a nice application of the pigeonhole principle.

4. Triangle treats

The first systematic study of mutual relationships between well-known triangle centers was made in 1767 by Euler [**7**], who showed that the circumcenter O, centroid G, and orthocenter H are collinear, with G dividing the segment OH into the ratio $1:2$. This fundamental property of triangles had been overlooked by the thousands of geometers who preceded him, from Euclid to Archimedes to Heron. In the century that followed, geometry experienced a kind of renaissance, and mathematicians had uncovered many curious new properties of the triangle [**9**]. William Dunham observed [**3**]: "If the Greeks gave us the Golden Age of geometry, then the century after Euler may well be regarded as a Silver Age." Early in the nineteenth century Brianchon and Poncelet discovered the nine-point circle whose center N is at the mid point of OH and whose radius is half the circumradius. A recent *Monthly* problem revisited these topics.
AMM 10796 [2001, 569]. *Proposed by Floor van Lamoen, The Netherlands.* Let ABC be a triangle, and let the feet of the altitudes dropped from A, B, C be A', B', C', respectively. Show that the Euler lines of triangles $AB'C'$, $A'BC'$, $A'B'C$ concur at a point on the nine-point circle of ABC.

Euler also found a number of expressions for the distances between triangle centers [**7**]. One of which is $OI^2 = R(R - 2r)$, where R is the circumradius and r is the inradius. This yields Euler inequality $R \geq 2r$. In the last century, geometric inequalities have attracted the attentions of many mathematicians [**1**], [**11**], among whom was Paul Erdös, another one of the best problem solvers of all times.
AMM 3740 [1937, 252-254]. *Proposed by Paul Erdös.* From a point O inside a triangle ABC perpendiculars OP, OQ, OR are drawn to its sides. Prove that
$OA + OB + OC \geq 2(OP + OQ + OR)$.
Given below are some other gems appeared recently in various problem sections.
MG 85.G [2001, 330]. *Proposed by Ho-joo Lee, South Korea.* From a point O inside a triangle ABC perpendiculars OP, OQ, OR are drawn to its sides BC, CA, AB

respectively. Prove that
$OA \cdot OB + OB \cdot OC + OC \cdot OA \geq 2(OA \cdot OP + OB \cdot OQ + OC \cdot OR)$.
MM 1620 [2001, 154]. *Proposed by Mihàly Bencze, Romania.* In $\triangle ABC$, let $a = BC$, $b = CA$, and $c = AB$. Let m_a, m_b, and m_c be the length of the medians from A, B, and C, let $s = \frac{1}{2}(a + b + c)$, and let R be the circumradius of $\triangle ABC$. Prove that
$\max\{am_a, bm_b, cm_c\} \leq sR$.
Crux 2662 [2001, 337]. *Proposed by Christopher J. Bradley, Clifton College, UK.* Suppose that $\triangle ABC$ is acute angled, has inradius r and has area Δ. Prove that
$(\sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C})^2 \leq \frac{\Delta}{r^2}$.
Crux 2628 [2001, 214]. *Proposed by Victor Oxman, University of Haifa, Israel.* Four points X, Y, Z, W are taken inside or on triangle ABC. Prove that there exists a set of three of these points such that the area of the triangle formed by them is less than $3/8$ of the area of the given triangle.

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ABBREVIATIONS
AMM: *The American Mathematical Monthly*
Crux: *Crux Mathematicorum with Mathematical Mayhem*
MG: *The Mathematical Gazette*
MM: *Mathematics Magazine*
PoW: *Problem of the Week* at Polk Community College

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