My Erdös Number Is $\sqrt{-1}$

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Please ignore the title. It's just a luring device to get you to my talk. A more appropriate title should be:

Problem-Solving Strategies, Illustrated by Challenging Integrals and Inequalities from Elementary Mathematics

Example 1. A Trigonometric Integral

Problem 667. Proposed by Mulatu Lemma, Savannah State University, GA, *College Mathematical Journal*, v.32 n.1 (Jan./2001), p.62-63

Find the value of the integral $\int_0^\pi \frac{2+2\cos x-\cos(n-1)x-2\cos nx-\cos(n+1)x}{1-\cos 2x}dx$ as a function of n, where n is a nonnegative integer.

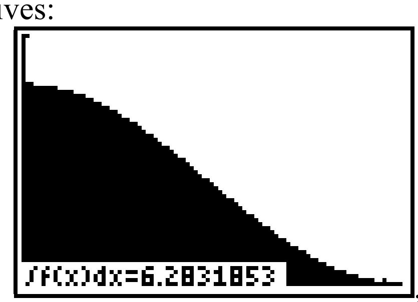
Let $f(n)$ be the given integral.

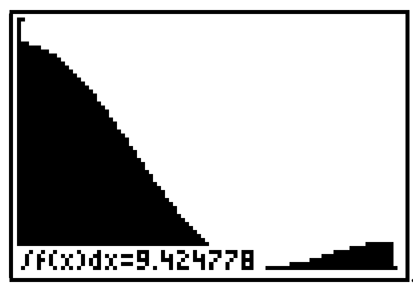
Srategy 1. Collect data

$f(0) = 0$; $f(1) = \pi$; and $f(2)$ is a little difficult already.

Strategy 2. Use calculator/computer

TI-83 gives:

$f(2) =$ 

, and $f(3) =$ 

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Strategy 3. Conjecture

$f(n) = n\pi$?

Direct proof is difficult and messy.

Strategy 4. View from different perspectives (*New Yorker*, Nov.6/2000, p.56)

It's a problem of integration. Or, is it? Wait, $n\pi$ is also an arithmetic sequence!

Can we prove $f(n+1) - f(n) = \pi$? Still difficult.

How about $f(n+1) - f(n) = f(n) - f(n-1)$?

Aha! There are three consecutive integers in the integral, so there is a good chance that $f(n+1) - 2f(n) + f(n-1)$ is easy. Indeed, it easily simplifies to $\int_0^\pi 2\cos(nx)dx = 0$.

In retrospect, we have used another important strategy:

Strategy 5. Work backward

This example also illustrates an interesting phenomena:

Strategy 6. "Inventor's paradox" (G. Pólya, *How to Solve It*, Princeton, 1988, p.121-122): The more ambitious plan may have more chances of success.

$f(2)$, $f(3)$, or $f(2001)$ may be difficult individually. But $f(n)$ is easy.

The paradox can also be illustrated by another problem I posed to my students at PCC:

Problem of the Week. Can $f(x) = \frac{1}{x^4 - x}$ be written as the sum of an even and an odd function? (I. M. Gelfand et al, *Functions and Graphs*, Birkhäuser, 1990, p.90)

Example 2. A Cubic Spherical Integral

Problem 10771. Proposed by Mowaffaq Hajja and Peter Walker, American University of Shajah, U. A. E., *American Mathematical Monthly*, v.107 n.10 (Dec./2000), p.954-955
Evaluate $\int_0^1 \int_0^1 \int_0^1 \frac{1}{(1+u^2+v^2+w^2)^2} dudvdw$.
Experiment with cylindrical coordinates, spherical coordinates, series expansion, geometrical and physical interpretations, etc. No luck!
Wait, we haven't used technology (Strategy 2). The more powerful TI-89 from one of my students gives the approximate answer: 0.3084251375.
Yeah right, that helps!
Strategy 6. "Take counsel of your pillow" (G. Pólya, p.198):
Give your problem a rest after a worthwhile effort.
Two weeks passed. Ready to give up.
Strategy 7. Never admit defeat
Never? You must be kidding!
OK, you win.
Modified Strategy 7. Never admit defeat *too easily*
Revisit the scratch papers, in cylindrical coordinates, the integral $I = 2\int_0^1 \int_0^{\pi/4} \int_0^{1/\cos\theta} \frac{1}{(1+r^2+w^2)^2} rdrd\theta dw = \int_0^1 \int_0^{\pi/4} [\frac{1}{1+w^2} - \frac{\cos^2\theta}{1+(1+w^2)\cos^2\theta}] d\theta dw = \int_0^1 \int_0^{\pi/4} \frac{1}{(1+w^2)[1+(1+w^2)\cos^2\theta]} d\theta dw$.
In desperation, integrate $\int_0^1 \int_0^{\pi/4} \frac{1}{1+w^2} d\theta dw$ for partial credit and get $\pi^2/16 \approx 0.6168502751$. Surprise, it's twice that number from TI-89 earlier!
Aha! We have a "boomerang integral" here:
Let θ = arctan$x$, then $d\theta = \frac{1}{1+x^2} dx$ and $\cos^2\theta = \frac{1}{1+x^2}$. Then
$\int_0^1 \int_0^{\pi/4} \frac{\cos^2\theta}{1+(1+w^2)\cos^2\theta} d\theta dw = \int_0^1 \int_0^1 \frac{1}{(1+x^2)[(1+x^2)+(1+w^2)]} dxdw$
$= \int_0^1 \int_0^1 \frac{1}{(1+w^2)[(1+x^2)+(1+w^2)]} dxdw = \int_0^1 \int_0^{\pi/4} \frac{1}{(1+w^2)[1+(1+w^2)\cos^2\theta]} d\theta dw$.
Hence, $I = \pi^2/32$.
If you think that integrals are after all "trivial" calculus problems, then try the next one.
Problem of the Millennium. Evaluate $\int_0^1 \int_0^1 \int_0^1 \frac{1}{1-uvw} dudvdw$. (M. S. Cohen et al, *Student Research Projects in Calculus*, MAA, 1991, p.196-197)
Yes, it equals $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which was first studied by Euler in 1736 and is still an open problem. (Euler discovered $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ in 1731, at the age of 24. For a wonderful account of the story, see W. Dunham, *Euler-The Master of Us All*, MAA, 1999, p.39-60)

Example 3. Convex Logarithms

Problem 1597. Proposed by Constantin P. Niculescu, University of Craiova, Romania, *Mathematics Magazine*, v.73 n.2 (Apr./2000), p.157

For every $x,\ y \in (0, \sqrt{\pi/2})$ with $x \neq y$, prove that
$\ln^2\frac{1+\sin xy}{1-\sin xy} < \ln\frac{1+\sin x^2}{1-\sin x^2} \cdot \ln\frac{1+\sin y^2}{1-\sin y^2}$.
Work backward (Strategy 5), and take "ln":
$2\ln\ln\frac{1+\sin xy}{1-\sin xy} < \ln\ln\frac{1+\sin x^2}{1-\sin x^2} + \ln\ln\frac{1+\sin y^2}{1-\sin y^2}$
Strategy 8. Think convexity when dealing with inequalities
Let $f(t)$ be ...
Strategy 9. ...let's see...hmmm... (R. L. Graham et al, *Concrete Mathematics*, Addison-Wesley, 1994, p.283),
yes, $\ln\ln\frac{1+\sin e^t}{1-\sin e^t}$. Then the desired inequality is $2f(\ln x + \ln y) < f(2\ln x) + f(2\ln y)$.
It's then easy to check $f''(t) > 0$.
An esthetic observation: "$\ln x$ bends the graph down, e^x bends it back".
Strategy 10. Have a beer after a nice solution (I. P. Pavlov)
(See also, P. Hadfield, Drink to think, *New Scientist*, Dec.9/2000, p.10)

Example 4. A Mean Inequality

Problem 10798. Proposed by Edward Neuman, Southern Illinois University, Illinois, *American Mathematical Monthly*, v.108 n.2 (Feb./2001), p.178
Given positive real numbers x and y, let A be their arithmetic mean, let G be their geometric mean, and let $L = (y - x)/(\ln y - \ln x)$ be their logarithmic mean. Prove that $A^L < G^A$ if both x and y are at least $e^{3/2}$ and that $A^L > G^A$ if both x and y are at most $e^{3/2}$.
Where on earth does $e^{3/2}$ come from?
Work backward and take "ln" (The beer strategy really works!):
$L\ln A < A\ln G$.
Strategy 11. Allow your pen to manipulate when your brain is blank
$\frac{y-x}{\ln y - \ln x}\ln\frac{x+y}{2} < \frac{x+y}{2}\ln\sqrt{xy}$;
$(\ln\frac{x+y}{2})/(\frac{x+y}{2}) < \frac{\ln y - \ln x}{y-x} \cdot \frac{\ln x + \ln y}{2} = \frac{1}{y-x} \cdot \frac{\ln^2 y - \ln^2 x}{2}$.
Think convexity again (Is that beer still working?) and let $f(t)$ to be, ...let's see...hmmm...(Strategy 9), yes,
$(\ln t)/t$. Then the result from the pen is $f(\frac{x+y}{2}) < \frac{1}{y-x}\int_x^y f(t)dt$.
Now, check the second derivative: $f''(t) = (2\ln t - 3)/t^3 = 0$ when $t = e^{3/2}$.
What a coincidence!

Example 5. An Erdös Connection

Please ignore this subtitle again. It's a device to wake you up.
Problem 10814. Proposed by Razvan Satnoianu, Oxford University, UK, *American Mathematical Monthly*, v.107 n.6 (Jun./2000), p.567
Let P be a point in the interior of a triangle. Let r, s, t be the distances from P to the vertices, and let x, y, z be the distances from P to the sides. Prove that for any $q \geq 1$,
a) $q^r + q^s + q^t + 3 \geq 2(q^x + q^y + q^z)$
b) $q^{s+t} + q^{t+r} + q^{r+s} + 6 \geq q^{2x} + q^{2y} + q^{2z} + 2(q^x + q^y + q^z)$.
Gee, you got to be kidding! Do I have to do this?
Don't give up too quickly (Strategy 7).

Think positively, think constructively, think beerwise! (Strategy 10)
Strategy 12. Draw a figure (G. Pólya, p.99)

Yeah right, that helps, wise guy!
Work backward? But how?
...
A week later, a bright idea! $q^x = e^{(\ln q)x} = \sum_{n=0}^{\infty} \frac{(\ln q)^n}{n!} x^n$. And a) becomes $3 + \sum_{n=0}^{\infty} \frac{(\ln q)^n}{n!}(r^n + s^n + t^n) \geq 2\sum_{n=0}^{\infty} \frac{(\ln q)^n}{n!}(x^n + y^n + z^n)$.
Strategy 13. Power to the power series!
Conjecture (Strategy 3): For $n \geq 1$, $r^n + s^n + t^n \geq 2(x^n + y^n + z^n)$?
It's too beautiful to be false.
Strategy 14. Recognize the sign of progess (G. Pólya, p.178)
Try some solid cases to make sure that the conjecture is very plausible. It is.
Now it's time to get our hands dirty for some geometry and trigonometry.

Let E and F be the feet of the perpendiculars from P upon CA and AB. Then $EF^2 = y^2 + z^2 - 2yz\cos\angle EPF = y^2 + z^2 + 2yz\cos A = y^2 + z^2 - 2yz\cos(B+C) = (y\sin C + z\sin B)^2 + (y\cos C - z\cos B)^2 \geq (y\sin C + z\sin B)^2$. Hence $r = \frac{AE}{\sin\angle APE} = \frac{AE}{\sin\angle AFE} = \frac{EF}{\sin A} \geq y\frac{\sin C}{\sin A} + z\frac{\sin B}{\sin A}$.
Strategy 15. Use symmetry

$$r \ \ge \ y\frac{\sin C}{\sin A} + z\frac{\sin B}{\sin A},$$
$$s \ \ge \ z\frac{\sin A}{\sin B} + x\frac{\sin C}{\sin B},$$
$$t \ \ge \ x\frac{\sin B}{\sin C} + y\frac{\sin A}{\sin C}.$$

And $\frac{\sin^n C}{\sin^n A} + \frac{\sin^n A}{\sin^n C} \ \ge \ 2$ by the AM-GM inequality.

Later I learned from the editorial comment to my solution that $r + s + t \ \ge \ 2(x + y + z)$ is the Erdös-Mordell inequality. It was first posed by Erdös in 1935 in *American Mathematical Monthly* and solved by L. J. Mordell in 1937.

Afterwords

We conclude with an important final strategy (G. Szegö's Preface to E. Rapaport, *Hungarian Problem Book I*, MAA, 1963, p.8):

Strategy 16. "We should not forget that the solution of any worthwhile problem very rarely comes to us easily and without hard work; it is rather the result of intellectual effort of days or weeks or months. Why should the young mind be willing to make this supreme effort? The explanation is probably the instinctive preference for certain values, that is, the attitude which rates intellectual effort and spiritual achievement higher than material advantage. Such a valuation can only be the result of a long cultural development of environment and public spirit which is difficult to accelerate by governmental aid or even by more intensive training in mathematics. The most effective means may consist of transmitting to the young mind the beauty of intellectual work and the feeling of satisfaction following a great and successful mental effort."