A Variant of Pascal's Triangle

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I. Introduction and Notation

It is the purpose of this paper to explore a variant of Pascal's triangle. This variant has the rule that every entry, denoted as $a_{n,k}$, where $a_{1,1}$=1, is calculated in a way such that $a_{n,k} = \sum_{i=1}^{k-1} a_{n-i,k-i} + \sum_{i=k}^{n-1} a_{i,k}$. This means that when these numbers are put into a triangular formation, every number is the sum of all the numbers above it in its two diagonals. The beginning of the triangle looks like the following:

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 2 && 2 && 2 &&& \\ && 4 && 5 && 5 && 4 && \\ & 8 && 12 && 14 && 12 && 8 & \\ 16 && 28 && 37 && 37 && 28 && 16 \end{array}$$

For example, 14 = 2 + 5 + 5 + 2.

The ***rows*** of the triangle will begin at n=1 and work down the triangle in increments of 1. The ***columns*** of the triangle point southwest (60^o from horizontal) and will begin at k=1, and proceed in increments of 1. For example, we would say that the leftmost number 12 would be in row 5 and in column 2. The ***anticolumns*** are the columns of the triangle pointing southeast that begin at k'=1, and proceed in increments of 1 moving diagonally from right to left. So, for example, the rightmost number 12 would be in row 5 and in anticolumn 2. Finally, ***Shallow diagonals*** point southwest (30^o from horizontal). For example, the third shallow diagonal consists of the elements $a_{3,1}$ = 2 and $a_{2,2}$ = 1.

II. Basic Lemmas & Theorems

Some interesting discoveries found within this triangle will now be discussed, and proved, a lot of which is done through induction. We begin with perhaps the most obvious result.

Theorem: The triangle is symmetrical (i.e.: $a_{n,k} = a_{n,n+1-k}$).

Proof: (By Induction) For n=3 and k=1, $a_{3,1} = 2 = a_{3,3+1-1} = a_{3,3}$. Now assume that $a_{n,k} = a_{n,n+1-k}$ for some n. Then we can write $\sum_{i=1}^{k-1} a_{n-i,k-i} + \sum_{i=k}^{n-1} a_{i,k} = \sum_{i=1}^{n-k} a_{n-i,n+1-k-i} + \sum_{i=n+1-k}^{n-1} a_{i,n+1-k}$ by definition. Further, $\sum_{i=1}^{k-1} a_{n+1-i,k-i} - \sum_{i=1}^{k-1} a_{n-i,k-i} + \sum_{i=k}^{n-1} a_{i,k} + a_{n,k} = \sum_{i=1}^{n-k} a_{n+1-i,n+1-k-i} + a_{n,n+1-k} - \sum_{i=n+1-k}^{n-1} a_{i,n+1-k} + \sum_{i=n+2-k}^{n} a_{i,n+2-k}$ by the induction hypothesis. This is the same as $\sum_{i=1}^{k-1} a_{n+1-i,k-i} + \sum_{i=k}^{n} a_{i,k} = \sum_{i=1}^{n+1-k} a_{n+1-i,n+2-k-i} + \sum_{i=n+2-k}^{n} a_{i,n+2-k}$ which means $a_{n+1,k} = a_{n+1,n+2-k}$. Hence, by induction, we conclude that the triangle is symmetrical.

We now develop some recursive and explicit formulas for calculating entries in the first three columns of the triangle. The first formula calculates the entries in the first column.

Lemma: $a_{n,1} = 2a_{n-1,1}$ for n$\geq$3.

Proof: By definition, $a_{n,1} = \sum_{i=1}^{n-1} a_{i,1} = a_{n-1,1} + \sum_{i=1}^{n-2} a_{i,1} = a_{n-1,1} + a_{n-1,1}$, since $a_{n-1,1} = \sum_{i=1}^{n-2} a_{i,1}$ by definition. Hence, $a_{n,1} = 2a_{n-1,1}$ for n$\geq$3.

An explicit formula for entries in the first column is now given.

Theorem: $a_{n,1} = 2^{n-2}$ for $n \geq 2$.

Proof: Since $a_{n,1} = 2a_{n-1,1}$, then the theory of difference equations implies $a_{n,1} = C\, 2^{n-1}$.

Solving for C when $a_{4,1} = 4$. We get C = ½, which means $a_{n,1} = 2^{n-2}$ for $n \geq 2$.

The following lemma is given to calculate the entries in the second column, and the next theorem is the corresponding explicit formula.

Lemma: $a_{n+1,2} = 2a_{n,2} + 2^{n-3}$ for $n \geq 3$.

Proof: We know $a_{n+1,2} = a_{n,1} + \sum_{i=2}^{n} a_{i,2}$ by definition. Further, $a_{n,1} = a_{n-1,1} + \sum_{i=1}^{n-2} a_{i,1}$ and $\sum_{i=2}^{n} a_{i,2}$ $= a_{n,2} + \sum_{i=2}^{n-1} a_{i,2}$. Thus, $a_{n+1,2} = a_{n-1,1} + a_{n,2} + \sum_{i=1}^{n-2} a_{i,1} + \sum_{i=2}^{n-1} a_{i,2}$. But, $\sum_{i=1}^{n-2} a_{i,1} = a_{n-1,1}$. So, $\sum_{i=1}^{n-2} a_{i,1} + \sum_{i=2}^{n-1} a_{i,2} = a_{n,2}$ by definition, which implies $a_{n+1,2} = a_{n-1,1} + 2a_{n,2}$. Therefore, by Theorem 1, $a_{n+1,2} = 2a_{n,2} + 2^{n-3}$ for $n \geq 3$.

Theorem: $a_{n,2} = (n+1)\, 2^{n-4}$ for $n \geq 4$.

Proof: The theory of difference equations implies $a_{n,2} = (A + Bn)(2^{n-4})$. Solving for A and B when $a_{3,2} = 2$ and $a_{4,2} = 5$, we get A=B=1. So, $a_{n,2} = (n+1)\, 2^{n-4}$.

Note: A different proof can be found in the appendix.

Finally, a recursive formula and corresponding explicit formula is given to calculate the entries in the third column, after which there does not appear to be a simple formula for calculating entries.

Lemma: $a_{n,3} = 2a_{n-1,3} + a_{n-2,2} + a_{n-2,1}$ for $n \geq 4$.

Proof: By definition, $a_{n,3} = \sum_{i=1}^{2} a_{n-i,3-i} + \sum_{i=3}^{n-1} a_{i,3} = a_{n-1,2} + a_{n-2,1} + \sum_{i=3}^{n-1} a_{i,3} = 2a_{n-2,2} + a_{n-3,1} + a_{n-2,1} + a_{n-1,3} + \sum_{i=3}^{n-2} a_{i,3} = a_{n-2,2} + a_{n-2,1} + a_{n-1,3} + a_{n-3,1} + a_{n-2,2} + \sum_{i=3}^{n-2} a_{i,3}$. But, $a_{n-1,3} = \sum_{i=1}^{2} a_{n-i,3-i} + \sum_{i=3}^{n-2} a_{i,3} = a_{n-3,1} + a_{n-2,2} + \sum_{i=3}^{n-2} a_{i,3}$ by definition. Thus, $a_{n,3} = a_{n-2,2} + a_{n-2,1} + a_{n-1,3} + a_{n-1,3} = 2a_{n-1,3} + a_{n-2,2} + a_{n-2,1}$ for $n \geq 4$.

Theorem: $a_{n,3} = (-4 + 7n + n^2)2^{n-7}$ for $n \geq 4$.

Proof: The theory of difference equations implies $a_{n,3} = \text{C}(2^{n-7}) + \text{D}n(2^{n-7}) + \text{E}n^2(2^{n-7})$.

Solving for C, D, and E when $a_{4,3} = 5$, $a_{5,3} = 14$, and $a_{6,3} = 37$, we get C= -4, D= 7, and E= 1, which gives $a_{n,3} = (-4 + 7n + n^2)2^{n-7}$.

Note: A different proof can be found in the appendix.

III. Other Lemmas & Theorems

These lemmas and theorems explain the behavior in the rows and shallow diagonals of the variant. Let S_n denote the sum of the entries in the n^{th} row, and D_n denote the sum of the entries in the n^{th} shallow diagonal.

Lemma: $S_n = 2S_{n-1} + 2S_{n-2} + 2S_{n-3} + \ldots + 2S_1$ for $n \geq 2$.

Proof: Choose an arbitrary element $a_{r,k}$. Consider the n^{th} row, where $n > r$. Then $a_{r,k}$ will appear in the formula for finding $a_{n,k}$ because it is in the same column. Further, $a_{r,k}$ will appear in the formula for finding $a_{n,k+n-r}$ because of the fact that $a_{r,k}$ is in its anticolumn. Hence, the arbitrary element $a_{r,k}$ is summed exactly twice in the row n for $1 \leq r \leq n-1$. And so, the sum of the elements in row n would be equal to twice the sum of the elements in all previous rows. Therefore, we can write $S_n = 2S_{n-1} + 2S_{n-2} + 2S_{n-3} + \ldots + 2S_1$ for $n \geq 2$.

Lemma: $S_n = 3S_{n-1}$ for $n \geq 3$.

Proof: It can be shown that $S_n = 2S_{n-1} + S_{n-1}$, and since $S_{n-1} = 2S_{n-2} + 2S_{n-3} + \ldots + 2S_1$. This means $S_n = 3S_{n-1}$.

The next theorem provides an explicit formula for calculating the sum of the entries in any given row.

Theorem: $S_n = (2/9)\ 3^n$ for n≥2 where $S_1 = 1$ and $S_2 = 2$.

Proof: From the theory of difference equations and lemma 4, $S_n = \mathrm{C} + \mathrm{D}3^n$. Since $S_2 = 2$ and $S_3 = 6$, solving for C and D, we get C = 0 and D = (2/9). Thus, $S_n = (2/9)\ 3^n$.

Note: A different proof can be found in the appendix.

The following theorem can be used to calculate any given entry in the triangle, and is used to prove the next couple theorems.

Theorem: $a_{n,k} = 2a_{n-1,k-1} + 2a_{n-1,k} - 3a_{n-2,k-1}$ for n ≥ 3.

Proof: We know $a_{n,k} = \sum_{i=1}^{k-1} a_{n-i,k-i} + \sum_{i=k}^{n-1} a_{i,k}$; $a_{n-1,k-1} = \sum_{i=1}^{k-2} a_{n-1-i,k-1-i} + \sum_{i=k-1}^{n-2} a_{i,k-1}$; $a_{n-1,k} = \sum_{i=1}^{k-1} a_{n-1-i,k-i} + \sum_{i=k}^{n-2} a_{i,k}$; and $a_{n-2,k-1} = \sum_{i=1}^{k-2} a_{n-2-i,k-1-i} + \sum_{i=k-1}^{n-3} a_{i,k-1}$. So, $a_{n,k} = a_{n-1,k-1} + \sum_{i=2}^{k-1} a_{n-i,k-i} + a_{n-1,k} + \sum_{i=k}^{n-2} a_{i,k}$, and $a_{n-1,k} = a_{n-2,k-1} + \sum_{i=2}^{k-1} a_{n-1-i,k-i} + \sum_{i=k}^{n-2} a_{i,k}$. Therefore, $a_{n,k} - a_{n-1,k} = a_{n-1,k-1} + a_{n-1,k} + \sum_{i=2}^{k-1} a_{n-i,k-i} - a_{n-2,k-1} - \sum_{i=2}^{k-1} a_{n-1-i,k-i}$. So,

$$a_{n,k} = a_{n-1,k-1} + 2a_{n-1,k} - a_{n-2,k-1} + \sum_{i=2}^{k-1} a_{n-i,k-i} - \sum_{i=2}^{k-1} a_{n-1-i,k-i} .$$

Further, $a_{n,k} - a_{n-1,k-1} = a_{n-1,k-1} + 2a_{n-1,k} - 2a_{n-2,k-1} - \sum_{i=2}^{k-1} a_{n-1-i,k-i} - \sum_{i=k-1}^{n-3} a_{i,k-1}$. So,

$$a_{n,k} = 2a_{n-1,k-1} + 2a_{n-1,k} - 2a_{n-2,k-1} - \sum_{i=2}^{k-1} a_{n-1-i,k-i} - \sum_{i=k-1}^{n-3} a_{i,k-1} .$$

Also, $a_{n,k} + a_{n-2,k-1} = 2a_{n-1,k-1} + 2a_{n-1,k} - 2a_{n-2,k-1}$. Hence, $a_{n,k} = 2a_{n-1,k-1} + 2a_{n-1,k} - 3a_{n-2,k-1}$.

With the next two theorems, we are able to calculate the sum of the entries in shallow diagonals.

Theorem: $D_n = 2D_{n-1} + 2D_{n-2} - 3D_{n-3}$ for n$\geq$5.

Proof: We know $a_{n,k} = 2a_{n-1,k-1} + 2a_{n-1,k} - 3a_{n-2,k-1}$ for n$\geq$2. If we let $a_{n,k}$ be in D_n, then $a_{n-1,k-1}$ is in D_{n-1}; $a_{n-1,k}$ is in D_{n-2}; and $a_{n-2,k-1}$ is in D_{n-3}. Consider $a_{r,k}$, an arbitrary element in D_n. If we let r = n+1-k, then $\sum_{k=1}^{\lceil n/2 \rceil} a_{n+1-k,k} = 2\sum_{k=1}^{\lceil n/2 \rceil} a_{n-k,k-1} + 2\sum_{k=1}^{\lceil n/2 \rceil} a_{n-k,k} - 3\sum_{k=1}^{\lceil n/2 \rceil} a_{n-k+1,k-1}$. It is important to note that in some of the summations there may exist terms in the form $a_{r,0}$ or $a_{r,r+1}$. These terms are actually equal to zero and hence do not affect the sum. Thus, $D_n = 2D_{n-1} + 2D_{n-2} - 3D_{n-3}$.

Theorem: $D_n = D_{n-1} + 3D_{n-2}$ for n$\geq$4.

Proof: (By Induction) For n=4, $D_4 = 4 + 2 = D_3 + 3D_2$. Now assume $D_n = D_{n-1} + 3D_{n-2}$ for some n. Also, recall $D_n = 2D_{n-1} + 2D_{n-2} - 3D_{n-3}$. So, $D_{n+1} = 2D_n + 2D_{n-1} - 3D_{n-2} = D_n + D_n + 2D_{n-1} - 3D_{n-2} = D_n + 2D_{n-1} + D_{n-1} + 3D_{n-2} - 3D_{n-2}$ by the induction hypothesis. Thus, $D_{n+1} = D_n + 3D_{n-1}$. Hence, by induction, we conclude $D_n = D_{n-1} + 3D_{n-2}$.

IV. The Triangle Mod 2 and 3

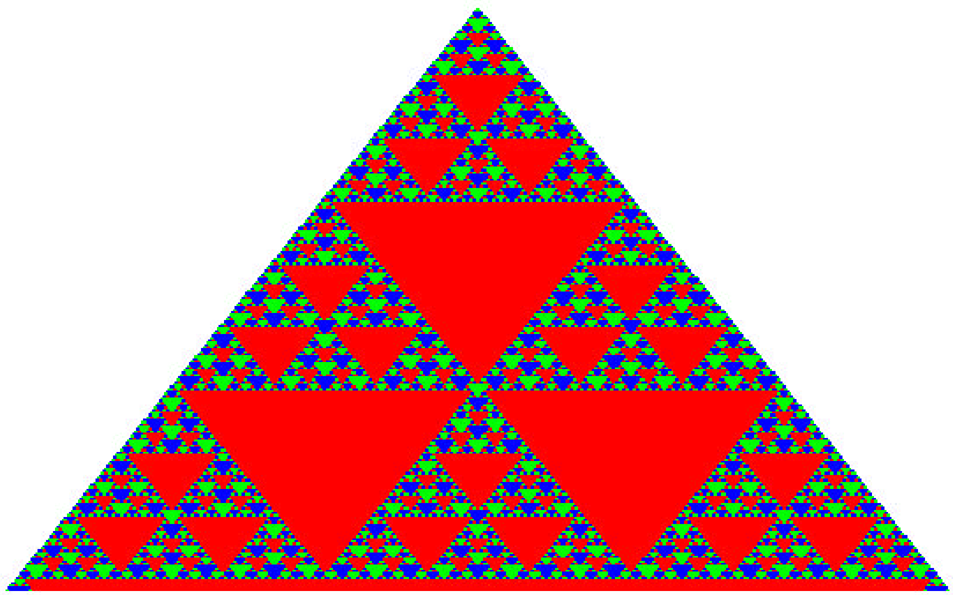
After examining the properties of the variant, we decided to explore the triangle mod 2 and 3, since Pascal's Triangle contained some interesting properties in its triangle mod 2 and 3. While Pascal's Triangle mod 2 displays a fractal structure, our variant does not, but it does display some intriguing patterns mod 3. A picture of the triangle mod 3, as well as the mentioned fractal patterns can be seen in the colored triangle mod 3 on the next two pages. The following theorem tells us that the triangle mod 2 does not display any fractal structure.

Theorem: The only odd numbers in the triangle are $a_{1,1}$ and entries of the form $a_{2n,k}$ and $a_{2n,k+1}$. These are consecutive numbers in the middle of every even numbered row.

Proof: (By induction on rows) Consider row 4. We see that row 3 has only even entries. Also, $a_{4,1}$ and $a_{4,4}$ are composed of 1+1+2+4 = 8, which is even. Further, $a_{4,2}$ and $a_{4,3}$ are composed of 1+2+2 = 5, which is odd. So, $a_{4,2}$ and $a_{4,3}$ are the only odd entries in row 4, and are in the center of the triangle. Thus, the hypothesis is true for row 3 and row 4. Now assume for some row r that all rows before and including it behave in the hypothesized fashion. Consider rows r+1 and r+2. Recall that $a_{n,k} = 2a_{n-1,k-1} + 2a_{n-1,k} - 3a_{n-2,k-1}$. So, $a_{n,k}$ will be even if $a_{n,k} \equiv 0$ (mod 2). Thus, for row r+1, $a_{r+1,k} = 2a_{r,k-1} + 2a_{r,k} - 3a_{r-1,k-1}$. But $a_{r-1,k-1}$ is even for all entries in row r-1 by the induction hypothesis, therefore $a_{r+1,k}$ = 0+0–0 = 0 (mod 2). Hence, all entries in row r+1 are even. For row r+2 we know that $a_{r,k-1}$ will be even except for the two consecutive entries in the middle of the row. This means that all entries in row r+2 will be even except for

{1}
{1, 1}
{2, 2, 2}
{1, 2, 2, 1}
{2, 0, 2, 0, 2}
{1, 1, 1, 1, 1, 1}
{2, 1, 1, 1, 1, 1, 2}
{1, 0, 1, 1, 1, 1, 0, 1}
{2, 2, 2, 1, 1, 1, 2, 2, 2}
{1, 2, 2, 0, 1, 1, 0, 2, 2, 1}
{2, 0, 2, 1, 2, 1, 2, 1, 2, 0, 2}
{1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1}
{2, 1, 1, 2, 0, 0, 0, 0, 0, 2, 1, 1, 2}
{1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1}
{2, 2, 2, 2, 2, 2, 0, 0, 0, 2, 2, 2, 2, 2, 2}
{1, 2, 2, 2, 2, 2, 1, 0, 0, 1, 2, 2, 2, 2, 2, 1}
{2, 0, 2, 2, 2, 2, 0, 2, 0, 2, 0, 2, 2, 2, 2, 0, 2}
{1, 1, 1, 2, 2, 2, 1, 1, 1, 1, 1, 1, 2, 2, 2, 1, 1, 1}
{2, 1, 1, 0, 2, 2, 0, 1, 1, 1, 1, 1, 0, 2, 2, 0, 1, 1, 2}
{1, 0, 1, 2, 1, 2, 1, 2, 1, 1, 1, 1, 2, 1, 2, 1, 2, 1, 0, 1}
{2, 2, 2, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 2, 2, 2}
{1, 2, 2, 1, 0, 0, 0, 0, 0, 2, 1, 1, 2, 0, 0, 0, 0, 0, 1, 2, 2, 1}
{2, 0, 2, 0, 2, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 2, 0, 2, 0, 2}
{1, 1, 1, 1, 1, 1, 0, 0, 0, 2, 2, 2, 2, 2, 2, 0, 0, 0, 1, 1, 1, 1, 1, 1}
{2, 1, 1, 1, 1, 1, 2, 0, 0, 1, 2, 2, 2, 2, 2, 1, 0, 0, 2, 1, 1, 1, 1, 1, 2}
{1, 0, 1, 1, 1, 1, 0, 1, 0, 2, 0, 2, 2, 2, 2, 0, 2, 0, 1, 0, 1, 1, 1, 1, 0, 1}
{2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2}
{1, 2, 2, 0, 1, 1, 0, 2, 2, 0, 1, 1, 0, 2, 2, 0, 1, 1, 0, 2, 2, 0, 1, 1, 0, 2, 2, 1}
{2, 0, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 0, 2}
{1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1}

The Triangle Mod 3



The colored Triangle Mod 3 (red = 0, green = 1, blue = 2) the two consecutive entries in the middle, which will be congruent to 1 (mod 2), which implies that they are odd. Thus, we conclude that the induction hypothesis is true.

The remaining results allow us to understand the structure of the triangle mod 3, and how it behaves.

Theorem: $a_{n,1} \equiv 1$ (mod 3) if n is even and $a_{n,1} \equiv 2$ (mod 3) if n is odd.

Proof: (By induction) We know $a_{n,1} = 2^{n-2}$ for n > 2. So, for n = 3, $a_{3,1} \equiv 2$ (mod 3) and for n = 4, $a_{4,1} \equiv 1$ (mod 3). Now assume $a_{n,1} \equiv 1$ (mod 3) if n is even and $a_{n,1} \equiv 2$ (mod 3) if n is odd. Then consider $a_{n+1,1} = 2a_{n,1}$. If $a_{n,1} \equiv 1$ (mod 3), then $a_{n+1,1} \equiv 2$ (mod 3); and if $a_{n,1} \equiv 2$ (mod 3), then $a_{n+1,1} \equiv 1$ (mod 3). Thus, by induction, we conclude that if n is even, $a_{n,1} \equiv 1$ (mod 3) and if n is odd, $a_{n,1} \equiv 2$ (mod 3).

From the following two lemmas, we are able to determine the structure of the entire triangle mod 3.

Lemma: If $a_{n,k} \equiv a_{n,k+1}$ (mod 3), then $a_{n+1,k+1} \equiv a_{n,k}$ (mod 3).

Proof: Let $a_{n,k} \equiv 0$ and $a_{n,k+1} \equiv 0$ (mod 3). We know $a_{n+1,k+1} = 2a_{n,k} + 2a_{n,k+1} - 3a_{n-1,k}$. So, $a_{n+1,k+1} \equiv 2(0) + 2(0) – 0$ (mod 3) which means $a_{n+1,k+1} \equiv 0$ (mod 3). Let $a_{n,k} \equiv 1$ (mod 3) and $a_{n,k+1} \equiv 1$ (mod 3). Then we can write $a_{n+1,k+1} \equiv 2(1) + 2(1) – 0$ (mod 3), which means $a_{n+1,k+1} \equiv 1$ (mod 3). Finally, let $a_{n,k} \equiv 2$ and $a_{n,k+1} \equiv 2$ (mod 3). Then this means, $a_{n+1,k+1} \equiv 2(2) +2(2) – 0$ (mod 3). Hence, $a_{n+1,k+1} \equiv 2$ (mod 3).

Lemma: If $a_{n,k} \neq a_{n,k+1}$ (mod 3), then $a_{n,k} \neq a_{n+1,k+1} \neq a_{n,k+1}$ (mod 3).

Proof: Consider $a_{n,k} \equiv 1$ (mod 3) and $a_{n,k+1} \equiv 2$ (mod 3). We know $a_{n+1,k+1} = 2a_{n,k} + 2a_{n,k+1}$ - $3a_{n-1,k}$. So, $a_{n+1,k+1} \equiv$ 2(1) + 2(2)- 0 (mod 3). Thus, $a_{n+1,k+1} \equiv 0$ (mod 3). Now consider $a_{n,k} \equiv 1$ (mod 3) and $a_{n,k+1} \equiv 0$ (mod 3). Then $a_{n+1,k+1} \equiv$ 2(1) + 2(0)- 0 (mod 3). Hence, $a_{n+1,k+1} \equiv 2$ (mod 3). Finally, consider $a_{n,k} \equiv 0$ (mod 3) and $a_{n,k+1} \equiv 2$ (mod 3). This means $a_{n+1,k+1} \equiv$ 2(0) + 2(2)- 0 (mod 3). Thus, $a_{n+1,k+1} \equiv 1$ (mod 3).

In addition, we have two conjectures:

Conjecture: In the triangle mod 3, triangular formations composed of zeroes will begin to take form if and only if the row number is a multiple of 9.

Conjecture: In the triangle mod 3 where row 4 is now considered row 1, there will be only one triangular formation composed of zeroes if and only if the row number can be written as 3^n for $n \geq 2$.

V. Summary and Future Research

To conclude, the following are the properties that have been found:

Theorem: The triangle is symmetrical (i.e.: $a_{n,k} = a_{n,n+1-k}$ **).**

Lemma: $a_{n,1} = 2a_{n-1,1}$ for n ≥ 3.

Theorem: $a_{n,1} = 2^{n-2}$ for n$\geq$2.

Lemma: $a_{n+1,2} = 2a_{n,2} + 2^{n-3}$ for n$\geq$3.

Theorem: $a_{n,2}$ = (n+1) 2^{n-4} for n$\geq$4.

Lemma: $a_{n,3} = 2a_{n-1,3} + a_{n-2,2} + a_{n-2,1}$ for n$\geq$4.

Theorem: $a_{n,3} = (-4 + 7n + n^2)2^{n-7}$ for n$\geq$4.

Lemma: $S_n = 2S_{n-1} + 2S_{n-2} + 2S_{n-3} + \ldots + 2S_1$ for n$\geq$2.

Lemma: $S_n = 3S_{n-1}$ for n$\geq$3.

Theorem: S_n = (2/9) 3^n for n$\geq$2 where $S_1 = 1$ and $S_2 = 2$.

Theorem: $a_{n,k} = 2a_{n-1,k-1} + 2a_{n-1,k} - 3a_{n-2,k-1}$

Theorem: $D_n = 2D_{n-1} + 2D_{n-2} - 3D_{n-3}$ for n$\geq$5.

Theorem: $D_n = D_{n-1} + 3D_{n-2}$ for n$\geq$4.

Theorem: The only odd numbers in the triangle are $a_{1,1}$ and entries of the form $a_{2n,k}$ and $a_{2n,k+1}$. These are consecutive numbers in the middle of every even numbered row.

Theorem: $a_{n,1} \equiv 1 \pmod 3$ if n is even and $a_{n,1} \equiv 2 \pmod 3$ if n is odd.

Lemma: If $a_{n,k} \equiv a_{n,k+1} \pmod 3$, then $a_{n+1,k+1} \equiv a_{n,k} \pmod 3$.

Lemma: If $a_{n,k} \neq a_{n,k+1} \pmod 3$, then $a_{n,k} \neq a_{n+1,k+1} \neq a_{n,k+1} \pmod 3$.

Conjecture: In the triangle mod 3, triangular formations composed of zeroes will begin to take form if and only if the row number is a multiple of 9.

Conjecture: In the triangle mod 3 where row 4 is now considered row 1, there will be only one triangular formation composed of zeroes if and only if the row number can be written as 3^n for $n \geq 2$.

A good avenue to take in continuing this research would be to 1) prove the conjectures, 2) search for ways in which the triangle may be useful in various fields of mathematics, as Pascal's triangle is, and 3) continue drawing parallels between Pascal's triangle and the variant. I do not necessarily know what these may entail, but they seem like a logical next step.

VI. Appendix

Some results were proven initially by induction and then a more concise proof was discovered at a later time. The following alternative proofs are the ones that were discovered first during the research.

Theorem: $a_{n,2}$ = (n+1) 2^{n-4} for n≥4.

Proof: (By Induction) When n = 4, $a_{4,2}$ = 2 (2) + 2^0 = 5 = (4+1) 2^0. Now assume that $a_{n,2}$ = (n+1) 2^{n-4} is true for some n. Since $a_{n+1,2} = 2a_{n,2} + 2^{n-3}$ we have $a_{n+1,2}$ = 2((n+1) 2^{n-4}) + 2^{n-3} = (n+2) 2^{n-3}, and we conclude by induction that $a_{n,2}$ = (n+1) 2^{n-4} for n≥4.

Theorem: $a_{n,3} = (-4 + 7n + n^2)2^{n-7}$ for n≥4.

Proof: (By Induction) First, for n=5, $a_{5,3} = 2a_{4,3} + a_{3,2} + a_{3,1}$ = 2(5) + 2 + 2 = 14 = (-4 + 7(5) + 5^2) 2^{-2}. Now assume that $a_{n,3} = (-4 + 7n + n^2)2^{n-7}$ is true for some n. Proceeding inductively,

$a_{n+1,3} = 2a_{n,3} + a_{n-1,2} + a_{n-1,1} \Rightarrow a_{n+1,3} = (-4 + 7n + n^2)2^{n-6} + n2^{n-5} + 2^{n-3} \Rightarrow a_{n+1,3} =$

$2^{n-3}[(-4 + 7n + n^2)2^{-3} + n2^{-2} + 1] \Rightarrow 2^{n-6}\,2^3\,[(1/2) + (9n/8) + (n^2/8)] \Rightarrow (4 + 9n + n^2)\,2^{n-6}$,

and we conclude by induction that $a_{n,3} = (-4 + 7n + n^2)2^{n-7}$ for n≥4.

Theorem: S_n = (2/9) 3^n for n≥2 where S_1 = 1 and S_2 = 2.

Proof: (By Induction) First, for n=3, $S_3 = 3S_2$ = 3(2) = 6 = (2/9) 3^3. Now assume that S_n = (2/9) 3^n for some n. So, $S_{n+1} = 3S_n \Rightarrow 3(2/9)\,3^n \Rightarrow S_{n+1} = (2/9)\,3^{n+1}$. Thus, by induction, S_{n+1} = (2/9) 3^{n+1} for n≥2.