SPECTRAL GEOMETRY, THE POLYA CONJECTURE, AND DIFFUSIONS

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Abstract. The purpose of this note is to define a collection of invariants associated to smoothly bounded domains in Riemannian manifolds and to study their relationship to well known geometric objects. We review a number of comparison type results, including an analog of the Polya conjecture. We study the relationship between our invariants and Dirichlet spectrum. The note is expository.

1: Introduction

There is a natural relationship between diffusion processes on a Riemannian manifold and the geometry of the underlying manifold. This relationship can be seen as arising via the association of a partial differential operator to each diffusion (the infinitesimal generator of the diffusion processes) on the one hand, and the connection between the geometry of the manifold and certain natural partial differential operators (the Laplace-Beltrami operator), on the other. The purpose of this note is to investigate a number of results which illustrate the nature of this connection. We are particularly interested in the connection between the Dirichlet spectrum (cf section 2) and a collection of probabilistic objects constructed using a natural diffusion (cf section 3). To introduce the objects of interest and to illustrate how they are "typically" related to the geometry of the underlying manifold, we begin by focussing on a theorem of comparison geometry:

Theorem 1.1 (cf [M1]). *Let (M,g) be a Riemannian manifold which admits an isoperimetric condition with constant curvature comparison space form M_κ. Let X_t be Brownian motion on M and, given $D \subset M$, suppose that $\tau = \tau_D = \inf\{t \geq 0 : X_t \notin D\}$ is the first exit time of X_t from D. Then, for all smoothly bounded domains D with compact closure, for all natural numbers n, if $vol_g(D) = v$, then*

$$\int_D E_x[\tau_D^n]dg \leq \int_B E_x[\tau_B^n]dg_\kappa \tag{1.1}$$

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where $B \subset M_\kappa$ is a geodesic ball of volume v, E_x denotes the expected value with respect to the measure charging Brownian paths beginning at x, and dg (respectively, dg_κ) is the volume form associated to the metric g (respectively, g_κ).

The goal of this report is to give some indication of why this result is useful/interesting, to give some motivation for the result, to give a few related results which suggest the existence of deep structure, and to sketch directions for future work. To accomplish this goal, the remainder of the note is broken into three parts. Section 2 is an introduction to the material which includes a discussion of the foundations of spectral and comparison geometry. Section 3 outlines the elementary probability and analysis required to understand the statement of the Theorem 1.1, as well as a number of additional theorems whose statement are natural given the introductory material. Finally, in section 3 we provide a number of results which describe the relationship between Dirichlet spectrum and our invariants. These results are most striking in the context of graphs and their associated Laplace operators. We develop graph theoretic material to the extent that it is needed for the presentation of our theorems.

2: Background Analysis and Geometry

Let $(x_1, x_2, \ldots, x_n)$ be standard coordinates on $\mathbb{R}^n$ with the usual smooth structure. The Laplace operator acting on smooth functions on $\mathbb{R}^n$ is given by

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Given a smoothly bounded domain $D \subset \mathbb{R}^n$ the Laplace operator restricts to act on smooth functions on D. If D has compact closure and we restrict the domain of the Laplace operator to functions which vanish at the boundary, the Laplace operator is self-adjoint (with respect to the usual inner product given by integration). The *Dirichlet spectrum* of the domain D, denoted $\mathrm{spec}(D)$, is the collection of corresponding eigenvalues of the Laplace operator. That is, $\mathrm{spec}(D)$ consists of those real numbers, λ, for which we can find a smooth solution of the boundary value problem

$$\begin{aligned} \Delta f + \lambda f &= 0 \text{ on } D \\ f &= 0 \text{ on } \partial D. \end{aligned} \tag{2.1}$$

When the domain D has compact closure, the spectrum is known to be discrete and positive. We will write

$$\mathrm{spec}(D) = \{\lambda_n : \ (2.1) \text{ has a solution with } \lambda = \lambda_n\}.$$

Classical spectral geometry arose as an attempt to clarify the relationship between the geometry of a Euclidean domain D and its associated spectrum, $\mathrm{spec}(D)$. To see why there should be an interesting relationship, note that there is a coordinate invariant description of the Laplace operator: if $d : C^\infty(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ denotes the exterior derivative mapping smooth functions to one-forms and $d^* : \Omega^1(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is the adjoint of d (with respect to the Euclidean metric), then the Laplace operator is given by

$$\Delta = d^* d.$$

There are two important things to realize about this formulation:

(1) The invariant definition depends on the smooth structure and on the metric, and hence the spectrum depends (only) on the smooth structure and the metric. In particular, the spectrum of a Euclidean domain is invariant under Euclidean motions.
(2) The invariant definition allows us to associate a Dirichlet spectrum to domains in arbitrary Riemannian manifolds (as well as to compact Riemannian manifolds without boundary).

Thus, we can formulate a fundamental problem in the category of Riemannian manifolds: What is the precise relationship between (Dirichlet) spectrum and geometry?

Perhaps the first example of a result of spectral geometry is the

Rayleigh conjecture:. *If D is a smoothly bounded domain in $\mathbb{R}^2$, then $vol(D) = v$ implies that $\lambda_1(D) \geq \lambda_1(B)$ where B is a disk of volume v.*

Rayleigh's conjecture was proven in the early twentieth century independently by Faber and by Krahn, who both noted that the key geometric property needed to establish this result was the classical *isoperimetric condition for $\mathbb{R}^2$*: Amongst plane domains of a given area v, the circle has minimal boundary length. Their results are a considerable generalization of the Rayleigh conjecture to the case of Riemannian manifolds which admit an analog of the classical isoperimetric condition. More precisely,

Definition. *Suppose that (M, g) is a Riemannian manifold. For $\kappa \in \mathbb{R}$, let M_κ be a constant curvature space form with curvature κ (if $\kappa > 0$, M_κ is a Euclidean sphere, if $\kappa = 0$, M_κ is a Euclidean space, if $\kappa < 0$, M_κ is an hyperbolic space). We say that M admits an isopermimetric condition with constant curvature comparison space M_κ if there exists κ such that for every domain $D \subset M$,*

$$vol_g(D) = v \text{ implies that } area_g(\partial D) \geq area_{g_\kappa}(\partial B)$$

where $B \subset M_\kappa$ is a geodesic ball of volume v, and $area_g(\partial D)$ denotes the surface area induced by g on the boundary of D.

That $\mathbb{R}^2$ admits an isoperimetric condition with comparison space $\mathbb{R}^2$ is the classical isoperimetric result known to the Greeks and established by Steiner in the nineteenth century. That constant curvature space forms likewise admit isoperimetric conditions with comparion spaces given by themselves is a result almost as old. A good reference for isoperimetric material is [BZ].

With this definition and these examples, we can now concisely state the Faber-Krahn result:

Faber-Krahn Theorem. *Suppose that (M, g) is a Riemannian manifold admitting an isoperimetric condition with constant curvature comparison space form M_κ. Suppose that $D \subset M$ is a smoothly bounded domain. Then*

$$vol_g(D) = v \text{ implies } \lambda_1(D) \geq \lambda_1(B) \tag{2.2}$$

where $B \subset M_\kappa$ is a geodesic ball of volume v.

The Faber-Krahn theorem is a result representative of that branch of mathematics labelled *Comparison Geometry*. Roughly speaking, results of Comparison Geometry involve hypotheses on a Riemannian manifold which allow comparison with a model space (eg an isoperimetric condition), and results which are usually phrased in terms of comparison with a model (eg the right hand side of (2.2)). Our main result (Theorem 1.1), as well as a number of related theorems we will discuss, fall into this class of results. For a survey of Comparion Geometry see [CE].

The Faber-Krahn theorem is also an example of a *direct result of spectral geometry.* That is, given constraints on the geometry of the space in which we work (a smoothly bounded domain of volume v inside a Riemannian manifold admitting an isoperimetric condition), we deduce results about the spectrum of the underlying space, thus establishing a connection between the geometry of a space and the corresponding Dirichlet spectrum. There are a great many such results (cf [Be] for survey material and an extensive bibliography) establishing a number of deep connections between the geometry of an ambient space and corresponding spectral data. Conjectures and results in the other direction (ie *inverse spectral results*) in which one constrains spectral data and deduces geometric constraints on the ambient space are in large measure resposponsible for much of the interest in spectral geometry during the last 50 years. An early conjecture, that Dirichlet spectrum determined planar domains up to isometry, was popularized by Mark Kac's "Can you hear the shape of a drum?" formulation of the problem (cf [Ka1]). This formulation resulted in rapid progress and extended interest in the field.

There are a variety of natural generalizations of the Faber-Krahn theorem which play an important role in the history of spectral and comparison geometry. One such generalization is a conjecture of Polya for Euclidean spaces which says that the conclusion of the Faber-Krahn Theorem should hold for all Dirichlet eigenvalues. More precisely,

Polya Conjecture. *Suppose that D is a smoothly bounded domain in Euclidean space. Then*

$$vol(D) = v \text{ implies that } \lambda_n(D) \geq \lambda_n(B) \tag{2.3}$$

where B is a ball of volume v.

While Polya failed to establish the conjecture that carries his name (as has everyone else that has tried to establish the result), he did manage to establish a great number of related geometric results for Euclidean domains that arose in the context of mathematical physics (cf [PS]). Many of these results were obtained using *symmetrization arguments* introduced by Steiner in his attempt to establish the isoperimetric inequality and used by Faber and Krahn in settling the Rayleigh conjecture (for an excellent introduction to symmetrization in the context of isoperimetric inequalities see [Ba]).

One of the important problems settled by Polya arose in the context of the study of the resistance of homogenous elastic bodies to an applied torque. More precisely, suppose we are given a homogenous cylinder whose cross section is given by a planar domain D. Let $u_1 : D \to \mathbb{R}^2$ be the function defined as the solution to the Poisson problem

$$\begin{aligned} \Delta u_1 + 1 &= 0 \text{ on } D \\ u_1 &= 0 \text{ on } \partial D. \end{aligned} \tag{2.4}$$

The *torsional rigidity of the domain D* is defined to be the quantity

$$A(D) = \int_D u_1(x)dx$$

where dx is Lebesgue measure. Torsional rigidity gives a precise quantitative measure of the resistance of a homogeneous cylinder to torque [Se]. Polya established

St. Venant Torsion Conjecture. *Let D be a smoothly bounded domain in the Euclidean plane. Then*

$$vol(D) = v \text{ implies that } A(D) \leq A(B)$$

where B is a disk of volume v.

By now it should be clear that there is a relationship between the Polya conjecture, the St. Venant torsion conjecture, and Theorem 1.1. Formally, Theorem 1.1 (for the case of $M = \mathbb{R}^n$) is the analog of the Polya conjecture with average exit time moments replacing eigenvalues (the Euclidean result follows from some work of Aizenman-Simon [AS]). On the other hand, Theorem 1.1 (for the case of $M = \mathbb{R}^2$ and $n = 1$) is a restatement of the St. Venant Torsion Conjecture. To understand this latter statement requires a little probability.

3: Background Probability

To understand the relationship between Theorem 1.1 and The St. Venant Torsion Conjecture, we need to understand how it is that exit time moments (probabilistic objects) are related to Poisson problems (analytic objects). The key to understanding this connection is understanding how the temperature at points of a domain which is initially heated to a uniform value, evolves when the boundary of the domain is held at the constant temperature zero. To give an idea of how these connections arise, we examine a particularly simple model for heat flow along a uniform wire.

We begin by restricting our attention to a collection of equally spaced points along the real line. Fix $\delta_x > 0$ and write

$$\mathbb{R}_{\delta_x} = \{x \in \mathbb{R} : x = m\delta_x \text{ for some integer } m\}.$$

We were interested in investigating the motion of a particle which starts at $X_0 \in \mathbb{R}_{\delta_x}$ and moves either one unit to the right or one unit to the left every time unit δ_t, according to the outcome of the flip of a fair coin. Let Ω be the collection of all sequences of heads and tails. We can think of Ω as a probability space (Ω can be thought of as sequences of 0's and 1's giving binary expansions for elements of $[0, 1]$.). Let $Y_i : \Omega \to \{1, -1\}$ be the random variables defined by

$$Y_i(\omega) = \begin{cases} 1 & \text{if the } i\text{th flip is heads, ie } \omega_i = H \\ -1 & \text{if the } i\text{th flip is tails, ie } \omega_i = T \end{cases}$$

Note that the Y_i are independent and identically distributed. In addition, we can use the Y_i to keep track of the position of our particle. We denote the position of the particle after n units of time by X_n :

$$X_n = X_0 + \delta_x \sum_{i=1}^{n} Y_i$$

where X_0 is the initial position of the particle. Note that X_n, as a sum of i.i.d. random variables, is a random walk.

If we think of time as an independent variable and plot X_n as a function of time, interpolating between points, we get, for each element of Ω and each starting point X_0, a piecewise linear trajectory representing the motion of our random particle. We are interested in understanding which trajectories are likely to occur and what probabilities we should associate to the occurence of such trajectories. Stated otherwise, we are interested in the transition probabilities of the random walk X_n, given that we know the starting position. Note that these transition probabilites can be thought of as defining a measure on the collection of piecewise linear paths which charge paths which begin at X_0.

To determine the transition probabilities we set

$$P_y(k,n) = \text{Prob}(X_n = k\delta_x | X_0 = y)$$

and we note that in order for the particle to be at position $k\delta_x$ at time n, it must be within one unit of $k\delta_x$ at time $n-1$:

$$P_y(k,n) = \frac{1}{2}\left(P_y(k+1,n-1) + P_y(k-1,n-1)\right). \tag{3.2}$$

A little algebra gives

$$\frac{P_y(k,n) - P_y(k,n-1)}{\delta_t} = \frac{1}{2}\frac{\delta_x^2}{\delta_t}\frac{P_y(k+1,n-1) - 2P_y(k,n-1) + P_y(k-1,n-1)}{\delta_x^2}. \tag{3.3}$$

Note that the left hand side of (3.3) is a difference quotient involving only the time variable while the right hand side is a difference quotient involving only the space variable. We write

$$D_t P_y(k,n) = \frac{1}{2}\frac{\delta_x^2}{\delta_t} D_x^2 P_y(k,n). \tag{3.4}$$

If we assume that P_y arises as the restriction of a smooth function and we send δ_t and δ_x to zero in such a way that $\frac{\delta_x^2}{\delta_t}$ approaches a limit, say 1, then the equality in (3.4) converges to

$$\frac{\partial P_y}{\partial t} = \frac{1}{2}\frac{\partial^2 P_y}{\partial x^2}. \tag{3.5}$$

We conclude that the transition probabilities $P_y(k,n)$ satisfy a discrete version of the heat equation. Our computations also suggest that in the right "continuum limit", the transition probabilities for the random walk provide solutions for the one-dimensional heat equation. We develop this further.

We will associate to each random walk the polygonal path which its trajectory defines. Such paths are continuous. It is possible to show that, given the relationship $\frac{\delta_x^2}{\delta_t} = 1$, the paths have limits which are well defined *continuous* paths which will depend on a continuous (as opposed to discrete) parameter t. Sufficient diligence to detail leads to an understanding of what we mean by "random motion on the line." We offer the following working definition:

We will say that a particle moves randomly or undergoes Brownian motion if the trajectories of the particle are continuous paths which are given as limits of the trajectories for our simple random walk, together with a family of probability measures, $P_y,\ y \in \mathbb{R}$, which charge those paths beginning at the point y. We will write such trajectories as

$$X_t = \lim_{\delta_t, \delta_x \to 0} X_n$$

where the limit is understood to occur under the restriction $\frac{\delta_x^2}{\delta_t} = 1$.

To give a true definition of Brownian motion would require lots of machinery and would obscure the properties of the process in which we are interested. Those interested in the details of a rigorous construction should check any one of a number of modern references. Two such references close in spirit to the development given above are the book of Ito and McKean [IM] and/or the expository paper of Kac's [Ka2].

Since the trajectories for Brownian motion arise as limits of the trajectories of our random walks, we should be able to study probabilistic properties of Brownian paths by studying corresponding probabilistic properties of random walks and taking an appropriate limit. For example, assuming that the process starts at $y = 0$ we examine the probability that at time t the process is at position x (more accurately, we will estimate $P(X_t \in [x, x + dx])$ where dx is small). If δ_x and δ_t are small enough, we should be able to approximate this by $P_y(k, n)$ where $t \simeq n\delta_t$ and $x \simeq k\delta_x$. That is,

$$P_y(x, t) \simeq P_y(k, n)$$

when δ_x and δ_t are small and $t \simeq n\delta_t,\ x \simeq k\delta_x$.

Recall, X_n behaves like a binomial random variable of mean zero and variance n, and thus, we expect

$$P_y(k, n) \simeq \frac{1}{\sqrt{2\pi n}} e^{-\frac{k^2}{2n}}$$

This leads us to a guess as to how $P_y(x, t)$ behaves for general $x \in \mathbb{R}$. We expect that

$$P_y(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

thus giving further confirmation that this approach allows us to interpolate between probability and PDE. In fact, we can push things further: Suppose that $f : \mathbb{R} \to \mathbb{R}$ and let

$$\begin{aligned} u(k, n) &= E_{k\delta_x}[f(X_n)] \\ &= \sum_j f(j\delta_x) P_{k\delta_x}(j, n). \end{aligned} \tag{3.6}$$

A direct computation then gives that u satisfies

$$\begin{aligned} D_t u &= \frac{1}{2} D_x^2 u \\ u(k, 0) &= f(k\delta_x) \end{aligned} \tag{3.7}$$

Thus, we expect that

$$u(x,t) = E_x[f(X_t)]$$

satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \qquad (3.8)$$
$$u(x,0) = f(x)$$

Note that the strategy for solving the heat equation is the same in both the discrete and the continuous case: average over the correct collection of paths (either random walk paths or Brownian paths). We can solve other boundary value problems using similar averaging techniques. For example, suppose that $I = [a,b]$ is an interval with compact closure and let τ be the first exit time of Brownian motion from I :

$$\tau = \inf\{t \geq 0 : X_t \notin I\}.$$

Suppose that $g : I \to \mathbb{R}$ and define

$$u_g(x) = E_x\left[\int_0^\tau g(X_t)dt\right]. \qquad (3.9)$$

Then u solves the Poisson problem:

$$\frac{1}{2}\Delta u + g = 0 \text{ on } I \qquad (3.10)$$
$$u = \text{ on } \partial I.$$

In particular, if we take $g = 1$, we see that

$$u_1(x) = E_x[\tau]$$

satisfies

$$\frac{1}{2}\Delta u_1 + 1 = 0 \text{ on } I$$
$$u = 0 \text{ on } \partial I.$$

This explains the connection between the "St. Venant Torsion Problem on a line segement" and a special case of Theorem 1.1.

It should be clear that there are a variety of directions in which we can generalize these results. In particular, we could work on a lattice in $\mathbb{R}^n$ instead of a lattice in $\mathbb{R}$, moving to nearest neighbors in some random uniform fashion. Then the arguments sketched above provide an intuitive approach to Brownian motion in $\mathbb{R}^n$ and its relation to the solution of boundary value problems. In particular, suppose that X_t is Brownian motion in $\mathbb{R}^n$ and that P_x is the corresponding family of probability measures charging Brownian paths beginning at $x \in \mathbb{R}^n$. Suppose that $D \subset \mathbb{R}^n$ is a smoothly bounded domain with compact closure and that τ is the first exit time of X_t from D. Let u be the solution of the Poisson problem

$$\frac{1}{2}\Delta u_1 + 1 = 0 \text{ on } D$$
$$u = 0 \text{ on } \partial D.$$

Then $u(x) = E_x[\tau]$.

Similarly, suppose that $p(x,t)$ is the probabilty distribution function of τ :

$$p(x,t) = P_x(\tau \geq t). \tag{3.11}$$

then $p(x,t)$ satisfies

$$\begin{gathered} \frac{1}{2}\Delta p - \frac{\partial p}{\partial t} = 0 \text{ on } D \times (0,\infty) \\ \lim_{t\to 0} p(x,t) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{if } x \in \partial D \end{cases} \\ p(x,t) = 0 \text{ if } x \in \partial D. \end{gathered} \tag{3.12}$$

Let $\hat{p}(x,\lambda)$ be the Laplace transform of $p(x,t)$:

$$\hat{p}(x,\lambda) = \int_0^\infty p(x,t)e^{-\lambda t}dt. \tag{3.13}$$

Expanding $e^{-\lambda t}$ and using (3.13) we have

$$\hat{p}(x,\lambda) = \sum_{k=0}^\infty E_x[\tau^k]\frac{(-\lambda)^k}{k!}. \tag{3.14}$$

Thus, we see that $\hat{p}(x,\lambda)$ satisfies

$$\begin{gathered} \frac{1}{2}\Delta\hat{p} + \lambda\hat{p} = 0 \text{ on } D \times (0,\infty) \\ \hat{p}(x,\lambda) = 1 \text{ if } x \in \partial D. \end{gathered} \tag{3.15}$$

Letting $\lambda \to 0$ in (3.14) and (3.15) we conclude that if u_k solves

$$\begin{aligned} \frac{1}{2}\Delta u_k + k u_{k-1} &= 0 \text{ on } D \\ u_k &= 0 \text{ on } \partial D. \end{aligned} \tag{3.16}$$

then

$$u_k(x) = E_x[\tau^k]. \tag{3.17}$$

In particular, we have generated a candidate for "higher torsion" in the St. Venant Torsion Problem: integrals of the functions $u_k(x) = E_x[\tau^k]$. As a natural generalization of the Saint Venant Torsion Problem, we expect a result like Theorem 1.1 to hold and indeed this is the case (the proof, found in [M1], uses the rearrangement techniques of Steiner).

Observe that Theorem 1.1 suggests that the *Poisson spectrum* $\text{pspec}(D) = \{A_k(D)\}$ defined by

$$A_k(D) = \int_D E_x[\tau^k]dg \tag{3.18}$$

is closely tied to the geometry of D. Indeed, it's easy to see that (as in the case of the Dirichlet spectrum) each element of the sequence is invariant under the isometry group of the ambient space (this follows immediately from (3.16)-(3.18)) and in this sense, elements of pspec(D) are *geometric invariants* associated to the domain D. The Faber-Krahn Theorem, the Polya Conjecture, the Saint Venant Torsion Problem, and Theorem 1.1 suggest that there might be an interesting relationship between pspec(D) and spec(D). Roughly stated, the relationship is one of "reciprocal for comparison results." In the next section we will make this relationship precise in the context of graphs and graph Laplace operators, where it is fully understood. For the remainder of this section, we describe a number of comparison results for Dirichlet spectrum and provide the suggested analogs in the context of Poisson spectrum.

When sectional curvatures are bounded, we have the following theorem of Cheng [C1]:

Theorem (cf [C1]). *Suppose that M is a complete Riemannian manifold all of whose sectional curvatures are bounded above by a given constant κ. Let $p \in M$ and suppose that $\delta > 0$ is less than the injectivity radius of M at p. Let $D(p,\delta) \subset M$ be the geodesic ball centered at p of radius δ. Then*

$$\lambda_1(D(p,\delta)) \geq \lambda_1(B_\kappa)$$

where B is a geodesic ball of radius δ in the constant curvature space form M_κ.

We have

Theorem 2.1 (cf [M2]). *Suppose that M is a complete Riemannian manifold all of whose sectional curvatures are bounded above by a given constant κ. Let $p \in M$ and suppose that $\delta > 0$ is less than the injectivity radius of M at p. Let $D(p,\delta) \subset M$ be the geodesic ball centered at p of radius δ. Then, for all natural numbers m,*

$$A_m(D(p,\delta)) \leq A_m(B_\kappa)$$

where B is a geodesic ball of radius δ in the constant curvature space form M_κ.

Similarly, when Ricci curvature is bounded, there is another theorem of Cheng [C2]

Theorem (cf [C2]). *Suppose that M is a complete Riemannian manifold of dimension n with Ricci curvature satisfying $Ric(\xi,\xi) \geq \kappa(n-1)|\xi|^2$. Let $p \in M$ and suppose that $\delta > 0$ is less than the injectivity radius of M at p. Let $D(p,\delta) \subset M$ be the geodesic ball centered at p of radius δ. Then,*

$$\lambda_1(D(p,\delta)) \leq \lambda_1(B_\kappa)$$

where B is a geodesic ball of radius δ in the constant curvature space form M_κ.

Once again, there is an analogous result for Poisson spectrum:

Theorem 2.2 (cf [M2]). *Suppose that M is a complete Riemannian manifold of dimension n with Ricci curvature satisfying $Ric(\xi,\xi) \geq \kappa(n-1)|\xi|^2$. Let $p \in M$ and suppose that*

$\delta > 0$ is less than the injectivity radius of M at p. Let $D(p,\delta) \subset M$ be the geodesic ball centered at p of radius δ. Then, for all natural numbers m,

$$A_m(D(p,\delta)) \geq A_m(B_\kappa)$$

where B is a geodesic ball of radius δ in the constant curvature space form M_κ.

These results are meant to illustrate the "reciprocal relationship" mentioned above. We are now in a position to clarify precisely what form the "reciprocal relationship" takes - at least in the discrete case.

4: Relations Between Dirchlet Spectrum and Poisson Spectrum - The Discrete Case

In the previous section we sketched the construction of symmetric random walks on a lattice in Euclidean space and associated to such a construction a discrete version of the Laplace operator acting on functions on vertices of the lattice. This construction is a special case of a much more general construction involving graphs which are endowed with edge weightings and vertex weightings. More precisely, let $G = (V,E)$ be a connected oriented bidirected graph with vertex set V and edge set E. Given $e \in E$, we will represent e as an ordered pair $e = (t(e), h(e))$ where $t(e),\ h(e) \in V$.

We will denote by $C^0(G)$ the vector space of real valued functions on V and by $C^1(G)$ the vector space of real valued functions on E. There is a natural coboundary operator $d : C^0(G) \to C^1(G)$ defined by

$$df(e) = f(h(e)) - f(t(e)).$$

Let $C_0^0(G) \subset C^0(G)$ and $C_0^1(G) \subset C^1(G)$ be the subspaces consisting of those functions with compact support.

Let $W_V : V \to \mathbb{R}^+$ be a vertex weighting. Associated to W_V there is an inner product on $C_0^0(G)$ defined by

$$\langle f, g\rangle_V = \sum_{x\in V} f(x)g(x)W_V(x). \tag{4.1}$$

Similarly, a function $W_E : E \to \mathbb{R}^+$ defines an inner product $\langle\cdot\ ,\ \cdot\rangle_E$ on $C_0^1(G)$.

Given a pair of functions W_V and W_E as above, we call the ordered pair $W = (W_V, W_E)$ a weighting for G if

$$W_E(x,y)W_V(x) = W_E(y,x)W_V(y) \tag{4.2}$$

where $x, y \in V$ ((4.2) corresponds to a reversibility condition). If $\mathcal{O}$ is an orientation for G and W is a weighting, we call the triple $(G, \mathcal{O}, W)$ a *graph with geometry.*

Given a weighting W, we will denote by $d_W^* : C^1(G) \to C^0(G)$ the associated adjoint of the coboundary map d. We will denote by $\Delta : C^0(G) \to C^0(G)$ the (vertex) Laplacian, $\Delta = d_W^* d$ (we will not investigate the edge Laplacian dd_W^*).

Interest in applications involving discrete Laplace operators of the type defined above can be traced to Kirchoff [Ki] who modelled simple circuits as finite graphs with each edge corresponding to the conductance of a given circuit component (cf [DSn] for a survey of random walks and electrical networks). As Kirchoff established, it is possible to give graph theoretic formulations for the laws governing current behavior in a simple circuit (Kirchoff's laws for voltage and current, Ohm's law). Given a simple circuit (ie a graph with each edge weighted to represent the conductance of a given component and each vertex of weight one) and a given input current, it is possible to formulate the problem of finding the induced circuit current as a Dirichlet problem involving the edge Laplacian. In addition, there is a solution to this Dirichlet problem given by "energy minimization" (Thomson's Principle, cf [Bi]).

Since Kirchoff, the study of graph Laplacians has yielded a remarkable wealth of information in a variety of contexts (the references [Bi], [C3], [Lo] and references therein provide expository introductions to some of these applications). Among those fields where graphs and their associated discrete boundary value problems have found interesting applications are potential theory (cf [Bi], [Du], and references therein), spectral theory (cf [DS], [Ge], [C3]), differential geometry and global analysis (cf [Do], [F1], [F2], [V1], [V2]). We restrict our our the study of boundary value problems for the interior Laplace Operator on compact subgraphs of a given ambient graph. We are interested in the relationship between the Dirichlet spectrum and the Poisson spectrum for such boundary value problems for graph Laplacians.

Definition 1.1. *Let $(G, \mathcal{O}, W)$ be a graph with geometry and let D be a domain in G. Let spec(D) be the spectrum of the interior Laplace operator associated to D. We define the set spec$^*(D)$ by*

$$spec^*(D) = \{\lambda \in spec(D) : \langle \phi_\lambda, 1_{iD} \rangle_V \neq 0\} \tag{4.3}$$

where ϕ_λ is a normalized eigenvector associated to λ and 1_{iD} is the indicator function of the interior of D, denoted iD.

We emphasize that spec$^*(D)$ contains no information concerning spectral multiplicity; it is a subset of the real numbers consisting of those eigenvalues whose corresponding eigenspace projects nontrivially onto constant functions.

The next result indicates that pspec(D) contains any geometric information contained in spec$^*(D)$:

Theorem 4.1 (cf[MM1]). *Suppose that $(G, \mathcal{O}, W)$ is a graph with geometry and that D and D' are domains in G. Then, with the notation as in Definition 1.1,*

$$pspec(D) = pspec(D') \text{ implies } spec^*(D) = spec^*(D')$$

and we say that pspec(D) determines spec$^*(D)$.

To understand both why this theorem is true and why it is the best one can reasonably hope for, we return to the heat equation and a particular collection of invariants associated to the elements of pspec(D).

Recall, if D is a smoothly bounded domain in a Riemannian manifold M and $H(x,t)$ solves the heat equation with intial data

$$\begin{aligned} \Delta H &= \partial_t H \text{ on } D \times (0,\infty) \\ H(x,0) &= 1 \text{ on } D \\ H(y,t) &= 0 \text{ on } \partial D \times (0,\infty) \end{aligned} \tag{4.4}$$

then the heat content of D is the function $Q(t)$ defined by

$$Q(t) = \int_D H(x,t)dg \tag{4.5}$$

where dg is the metric density. We note that these definitions make perfect sense in the context of graphs with geometry.

There is a close connection between the heat content of D and the Poisson spectrum. From (3.12) it is clear that the heat content is the integral of the distribution function for the exit time of Brownian motion over the given domain. For each $x \in D$, it follows from the classical theory of moments that the distribution function $P_x(\tau \geq t)$ is determined by the functions $E_x[\tau^k]$ (cf [MM2]). We have:

Theorem 4.2 (cf [MM1], [MM2]). *Let M be a complete Riemannian manifold, $D \subset M$ a smoothly bounded domain with compact closure. Then pspec(D) determines the heat content of D. The same results hold in the context of graphs with geometry.*

By a theorem of Gilkey [Gi], the heat content $Q(t)$ given by (4.5) admits an asymptotic expansion for small t :

$$Q(t) \simeq \sum_{n=0}^{\infty} q_n t^{\frac{n}{2}} \tag{4.6}$$

where the coefficients q_n are given as integrals of metric invariants associated to D. The coefficients in (1.17) are sometimes referred to as the *heat content asymptotics.* Theorem 4.2 leads to the following result in the discrete setting (cf [MM1]),

Theorem 4.3 (cf [MM1]). *Suppose that $(G, \mathcal{O}, W)$ is a graph with geometry and that D is a domain of G. Suppose the heat content asymptotics of D are given by $\{q_n\}$. Then $\{q_n\}$ determines spec$^*(D)$. In addition, $\{q_n\}$ is determined by pspec(D), and determines pspec(D).*

The key to proving Theorem 4.3 involves an interesting Dirichlet series whose definition involves the manner in which constant functions decompose relative to the eigenspaces of the Dirichlet Laplacian. More precisely, define a *spectral partition of the volume* as follows:

Definition. *For $\lambda \in spec(D)$, let E_λ be the eigenspace associated to λ, and let $\{\phi_{\lambda,j} : 1 \leq j \leq dim(E_\lambda)\}$ be an orthonormal basis of E_λ. Let*

$$a_{\lambda,j} = \langle 1, \phi_{\lambda,j} \rangle_D \tag{4.7}$$

The collection $\{a_{\lambda,j}\}$ where λ varies over the Dirichlet spectrum is called a spectral partition of volume.

Note that the definition of a spectral partition of the volume depends on the choice of an eigenbasis. The norm of the projection of the function $f(x) = 1$ on a given eigenspace does not depend on the choice of a basis. Thus, the quantity

$$\hat{a}_\lambda^2 = \sum_{k=1}^{\text{mult}(\lambda)} a_{\lambda,j}^2$$

does not depend on choice of basis. For s a complex number we define a Dirichlet series by

$$\zeta_D(s) = \sum_{\lambda_j \in \text{spec}(D)} a_{\lambda_j}^2 \left(\frac{1}{\lambda_j}\right)^s. \tag{4.8}$$

In the context of graphs, the spectral data is finite and there is no question that the series converges. In addition, it is clear from (4.7) that $\zeta_D(s)$ does not depend on the choice of basis. Finally, it is clear from (4.8) that the sum in the definition could be taken over $\text{pspec}(D)$.

The following result gives a concise meaning to what we meant by "reciprocal relationship" in the previous section (the proof can be found in [MM1]):

Theorem 4.4 (cf [MM1]**).** *Let $(G, \mathcal{O}, W)$ be a graph with geometry, $D \subset G$ a domain with nonempty boundary. Let $\zeta_D(s)$ be defined as in* (4.8)*. Then, if m is a natural number,*

$$\zeta_D(m) = \frac{A_m(D)}{m!}$$
$$\zeta_D(-m) = (-1)^m m! q_m(D)$$

where $A_m(D)$ are the elements of the Poisson spectrum and $q_m(D)$ are the heat content asymptotics.

For the case of Riemannian manifolds, there are a number of results analogous to those of Theorem 4.3 and Theorem 4.4. Proofs can be found in [MM2].

References

[**AS**] M. Aizenman and B. Simon, *Brownian Motion and Harnack inequalities for Schrodinger operators*, Comm. Pure and Appl. vol35 (1982), 209-273.

[**Al**] D. Aldous, *Applications of random walks on finite graphs*, Selected Proceedings of the Sheffield Symposium on Applied Probability (Sheffield, 1989) volIMS Lecture Notes Monograph Ser., 18, Inst. Math. Statist., Hayward, CA (1991), 12-26.

[**Ba**] C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman Publishing, Marshfield Mass., 1980.

[**Be**] P. H. Bérard, *Spectral Geometry: Direct and Inverse Problems*, Springer LNMS vol1207, New York, N.Y., 1984.

[**Bi**] N. Biggs, *Algebraic potential theory on graphs*, Bull. London Math. Soc. vol29 (1998), 641-682.

[**BZ**] Y. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Springer Grundlehren 285, New York, N.Y., 1980.

[**C1**] S. Y. Cheng, *Eigenfunctions and eigenvalues of the Laplacian*, Amer. Math. Soc. Proc. Symp. Pure Math. Part II vol27 ((1975)), 185-193.

[**C2**] S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. vol143 ((1975)), 289-297.

[**C3**] F. R. K. Chung, *Spectral Graph Theory*, AMS CBMS Regional Conference Series in Mathematics, vol92, Providence, RI, 1997, pp. 12-26.

[**CE**] J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian Geometry*, North Holland Publ. Co., Amsterdam, 1975.

[**DS**] P. Diaconis and D. Stroock, *Geometric bounds for the eigenvalues of Markov chains*, Ann. Applied Prob. vol1 (1991), 36–61.

[**Do**] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans AMS vol284 (1984), 787–794.

[**DSn**] P. G. Doyle and J. L. Snell, *Random walks and electrical networks*, MAA Carus Monographs vol22, Washington, D.C., 1984.

[**Du**] R. Duffin, *Discrete potential theory*, Duke Math. J. vol20 (1953), 233–251.

[**F1**] R. Forman, *Difference operators, covering spaces and determinants*, Topology vol28 (1989), 413–438.

[**F2**] R. Forman, *Determinants and Laplacians on graphs*, Topology vol32 (1993), 35–46.

[**Gi**] P. Gilkey, *Heat content asymptotics*, In: Geometric aspects of partial differential equations (Roskilde, 1998), Contemp. Math. vol242 (1999), 125–133.

[**IM**] K. Ito and H. P. McKean, *Diffusion Processes and Their Sample Paths*, Springer Grundlehren 125, New York, N.Y., 1974.

[**Ka1**] M. Kac, *Can One hear the shape of a drum?*, Amer. Math, Monthly vol73 (1966).

[**Ka2**] M. Kac, *Random walk and the Theory of Brownian motion*, Amer. Math Monthly vol47 (1947).

[**Ki**] G. Kirchoff, *Über die Auflösung der Gleichungen auf Welche Man beider Untersuchen der Linearen Vertheilung Galvanischer Ströme Gefüft Wird*, Annalen der Physik und Chemie vol72 (1847), 495–508.

[**Lo**] L. Lovász, *Combinatorics*, Bolyai Society Mathematical Studies vol2, Hungary, 1993.

[**M1**] P. McDonald, *Isoperimetric conditions, Poisson problems and diffusions in Riemannian manifolds*, Potential Analysis (to appear).

[**M2**] P. McDonald, *Exit Time Moments and two Theorems of Cheng* (In preparation).

[**MM1**] P. McDonald and R. Meyers, *Diffusions On Graphs, Poisson Problems, and Spectral Geometry* (submitted).

[**MM2**] P. McDonald and R. Meyers, *Dirichlet Spectrum and Exit Time Moments of Brownian Motion* (preprint).

[**PS**] G. Polya and G. Szego, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, N.J., 1951.

[**Se**] J. Serrin, *A symmetry problem in potential theory*, Arch. Rat. Mech. and Anal. vol43 (1971), 304-3-18.

[**SH**] J. A. Shohat and J. D. Tamarkin, *The Problem Of Moments*, Amer. Math. Soc., New York, 1943.

[**Sp**] F. Spitzer, *Principles Of Random Walk*, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1964.

[**V1**] N. Varopolous, *Isoperimetric inequalities and Markov chains*, J. Funct. Anal. vol63 (1985), 215–239.

[**V2**] N. Varopolous, *Brownian motion and random walks on manifolds*, Ann. Inst. Fourier vol34 (1984), 243–269.

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