The Mathematics of Sudoku - Contest B*

1) Imagine a completely blank 9×9 grid (no givens). According to the rules of Sudoku, how many ways are there to validly fill in the top-left 3×3 block? $9! = 362,880$

2) Consider the puzzle below. How many ways can the remaining cells of the first row be validly filled in? $6! = 720$

1	2	3						
4	5	6						
7	8	9						

3) Suppose in the puzzle above, that the first row of the top-center 3×3 is also filled in and reads $4-5-6$. How many ways are there to complete the first three rows of this puzzle?

$(3!)^5 = 7,776$

4) Repeat question 3, but this time the first row of the top-center 3×3 reads $4-8-6$.

$18(3!)^4 = 23,328$

5) Given a completely blank 9×9 grid (no givens), how many ways are there to validly fill in the first three rows? $9! \cdot (12 \cdot 7,776 + 108 \cdot 23,328) = 9! \cdot (2,612,736) = 948,109,639,680$

6) Consider the below solved Sudoku puzzle. If you swap the top-center 3×3 block with the top-right 3×3 block, which permutations would be needed to retain a valid Sudoku grid? Swap the center 3×3 block with the right-center 3×3, and swap the bottom-center 3×3

with the bottom-right 3×3.

7) In Grid S, if you swapped the first two columns of the top-center 3×3 grid, which permutations would be needed to retain a valid Sudoku grid? Swap the first two columns of the center 3×3 block, and swap the first two columns of the bottom-center 3×3 block.

1	2	4	5	6	7	8	9	3
3	7	8	2	9	4	5	1	6
6	5	9	8	3	1	7	4	2
9	8	7	1	2	3	4	6	5
2	3	1	4	5	6	9	7	8
5	4	6	7	8	9	3	2	1
8	6	3	9	7	2	1	5	4
4	9	5	6	1	8	2	3	7
7	1	2	3	4	5	6	8	9

Figure 1: Grid S

8) How many distinct Sudoku grids do you think there are (educated guess)? Bertram Felgenhauser and Frazer Jarvis calculated this to be $6,670,903,752,021,072,936,960 \approx 6.6709 \times 10^{21}$.

9) Depending on the number of givens, an initial Sudoku puzzle may have more than one solution. How many of the different symbols $1, 2, 3, ..., 9$ would have to have initially appeared if the puzzle did have exactly one solution? Eight

10) Imagine rotating Grid S 90° clockwise. Do you still have a valid Sudoku grid? What other elementary transformations of Grid S would yield a new Sudoku grid? Yes. Elementary transformations are:

- clockwise rotation by 0°, 90°, 180°, or 270°
- reflection about the middle row, middle column, or about either diagonal

- relabeling the symbols $1, 2, 3, ..., 9$
- permuting rows within either the top, center, or bottom 3×9 block
- permuting columns within either the left, center, or right 9×3 block
- if viewing the grid as a large 3×3 of three "bands" and three "stacks," permuting the three bands, or permuting the three stacks (for example, question 6 ultimately swaps the center stack and the right stack).

11) Imagine the set of all elementary transformations of Grid S, together with any finite number of compositions of these transformations. Show that this set forms a group under the operation: composition of functions. Is the group abelian? This is a non-abelian group.

12) After having rotated Grid S $90°$ clockwise, find a relabeling of the numbers $\{1, 2, 3, \ldots, 9\}$ that gives you back Grid S.

$$1 \to 3, \quad 2 \to 6, \quad 3 \to 9, \quad 4 \to 2, \quad 5 \to 5, \quad 6 \to 8, \quad 7 \to 1, \quad 8 \to 4, \quad 9 \to 7$$

13) Let E be the set of all Sudoku grids T satisfying $T = \tau(S)$ for some transformation τ. Is every possible Sudoku grid contained in the set E? If not, how many of these types of sets E do you think there are (educated guess)? Ed Russell and Frazer Jarvis calculated this to be 5,472,730,538.

*The questions here are based on content from a web-tutorial called "The Math Behind Sudoku" found at http://www.math.cornell.edu/∼mec/Summer2009/Mahmood/Home.html