Chaotic Systems of Geodesics on Surfaces of Revolution

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28 April 2017

Abstract. We work on the problem of creating a surface embedded in $\mathbb{R}^3$ that has a chaotic system of geodesics, meaning all geodesics must be strongly unstable. Our surface will have two concentric parts connected by tubes, so we create a tube by designing a profile curve for a surface of revolution. The tube is the "danger" zone of the surface where geodesics may fail to be strongly unstable. We study surfaces of revolution in general, looking at the torus as an example, to build an understanding of how to test whether a geodesic is strongly unstable, then look at seven specific geodesics on our tube.

Introduction

In this paper we work on designing a tube that could be used as part of a surface in $\mathbb{R}^3$ with a chaotic system of geodesics. Every geodesic on the surface must be strongly unstable (see Definition 3.1) in order for the system to be chaotic. Donnay and Pugh [2] showed that we can make such a surface by connecting two concentric spheres with tubes if the number of tubes approaches infinity (see Figure 1). We want to use the same idea but with a small number of tubes. To do this, we imagine two concentric fire hydrant-like shapes (see Figure 2) that we will connect with tubes based on the the tube inside of a torus (see Figure 3). The purpose of using these hydrant shapes rather than spheres is that while spheres have positive curvature, the hydrants have negative curvature; we will see that negative curvature is necessary for strongly unstable geodesics.

We begin by studying surfaces of revolution in general, because tubes can be parametrized as surfaces of revolution. Since we want to base our tube on the torus, we will use it as a helpful and simple example of a surface of revolution. Instead of curving around to make the torus, however, we want our tube to stay more flat on the top to connect to the larger outside hydrant shape. We will create such a tube by designing a profile curve that gets rotated into a surface of revolution. Then we will study various geodesics on the tube to see if they are strongly unstable.

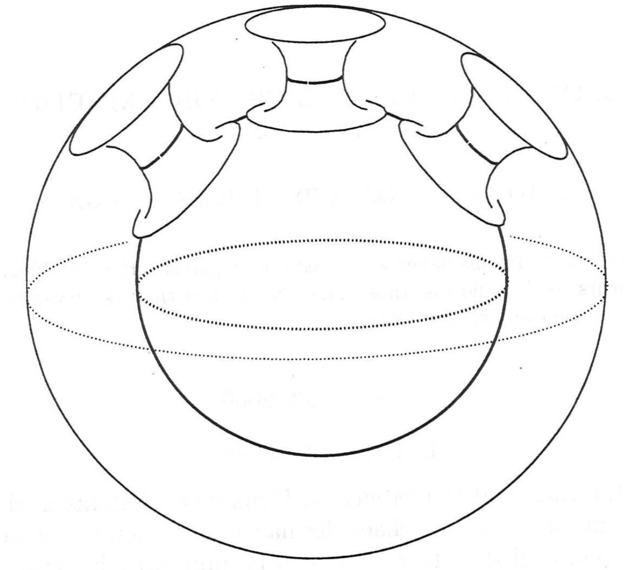

Figure 1: The surface made of two concentric circles connected by a number of tubes, created by Donnay and Pugh [2].

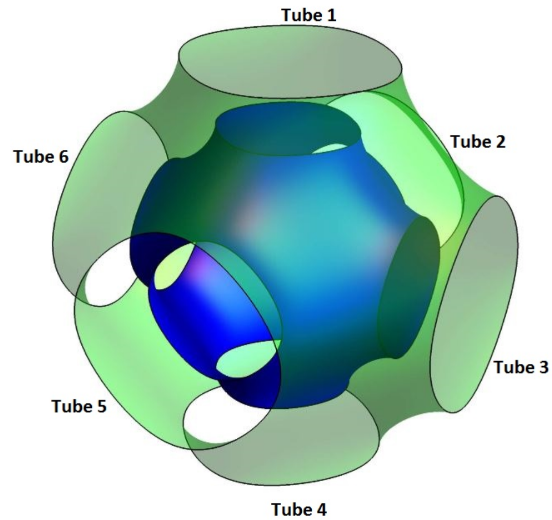

Figure 2: A proposed shape of two concentric "fire hydrants" that would be connected by six tubes.

1 Curvature of Surfaces of Revolution

1.1 Surfaces of Revolution

Since our goal is to create a tube and a tube is a surface of revolution, we start by defining and exploring surfaces of revolution.

Definition 1.1 (Surface of Revolution)**.** A surface of revolution is a surface created by rotating a plane curve in a circle. It can be written as a parametric function of the form $\mathbb{X}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$, where $(f(v), g(v))$ is the plane curve being rotated around angle u. We call $\alpha(v) = (f(v), g(v))$ the *profile curve.* The function $\mathbb{X}$ is known as a *chart* of a surface of revolution.

Remark 1. We will assume in general that profile curves for surfaces of revolution are parametrized by arc length, that is, $\|\alpha'(t)\| = 1$ for all t. Then

$$f'(t)^2 + g'(t)^2 = 1. \tag{1}$$

We will prove that the profile curve we end up using is actually parametrized by arc length in Proposition 4.1.

Example 1.1 (Torus)**.** The torus (see Figure 5) is a surface of revolution created with a circle as the profile curve. It is parametrized by the function

$$T(u, v) = ((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v)$$

where r is the radius of the profile curve circle and R is the radius measured from the axis of revolution to the center of the profile curve circle (see Figure 4).

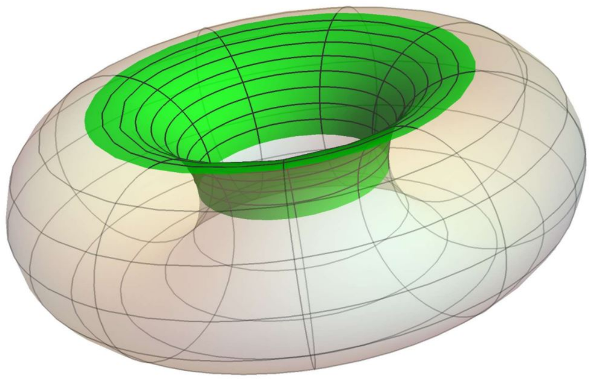

Figure 3: The inner tube part (green) of the torus (entire shape) that we based our tube on.

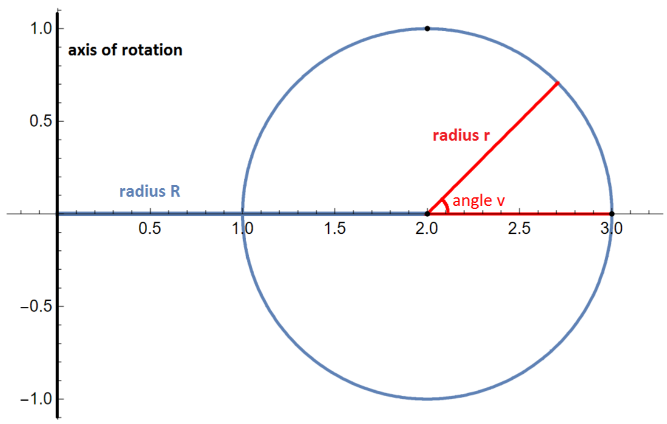

Figure 4: An image of a "cross-section" of the torus, showing the radius R, the profile circle, and how the angle v affects height on the circle.

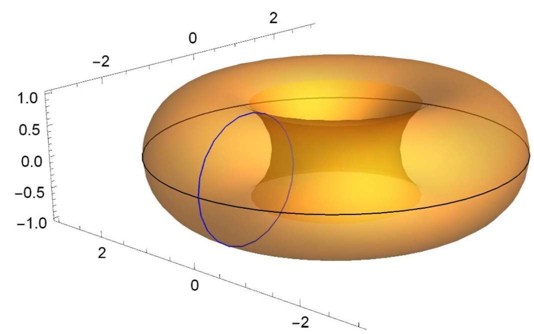

Figure 5: A torus with $R = 2, r = 1$. The blue circle is the profile curve.

1.2 Curvature

There are many types of curvature that can be calculated for curves and surfaces embedded in $\mathbb{R}^3$. We will discuss the Gaussian curvature as it is the type of curvature we will need further on for the Riccati and Jacobi equations. In general, curvature measures how sharply a surface or space curve bends. For instance, the Earth has a much smaller curvature than the ping-pong ball. This example makes sense: from our perspective on the surface of Earth, it looks flat, but we can see the roundness of a ping-pong ball.

The Gaussian curvature at a point on a surface is found by looking at two curves (called principal curves) that lie on the surface and intersect at that point. If at least one of two curves is locally a straight line at the point, then the surface has *zero* Gaussian curvature there. If the two curves bend in the same direction, like around a bowl, then the surface has *positive* Gaussian curvature at that point. If they bend in opposite directions like on a pringle, the surface has *negative* Gaussian curvature at that point (see Figure 6 for examples).

To define Gaussian curvature exactly we first need to define the following:

Definition 1.2 (Space Curvature)**.** Let $\alpha(s)$ be a space curve that is parametrized by arc length. Then $\kappa(s) = \|\alpha''(s)\|$ is the *curvature of the space curve* α at time s. We will call the curvature of a space curve "space curvature" hereafter.

Definition 1.3 (Normal Curvature)**.** Let $\alpha(s)$ be a curve along a surface S. Then the normal curvature $\kappa_n(s)$ of α at time s is the projection of $\alpha''(s)$ onto the normal to the surface at s.

Definition 1.4 (Principal Curvature)**.** Let S be a surface. Then the principal curvatures $k_1(p)$ and $k_2(p)$ at a point p on S are the maximum and minimum of the normal curvature taken along curves of every direction through p. The two curves that achieve the max and min normal curvature are called the *principal curves*.

Now we can define:

Definition 1.5 (Gaussian Curvature)**.** Let S be a surface. Then the Gaussian curvature K at a point p is $K_p = k_1(p)k_2(p)$.

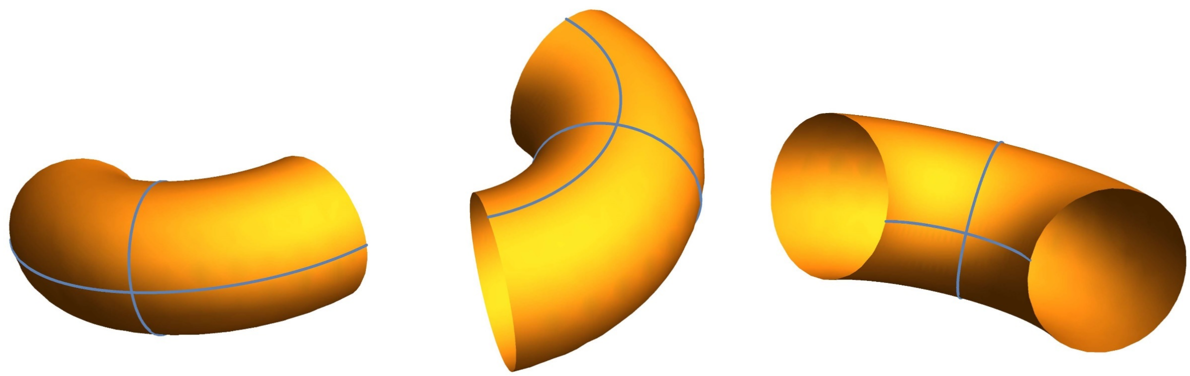

Figure 6: The principal curves are shown for positive, zero, and negative (left, center, right) curvature on a torus.

These definitions give us a way to find the Gaussian curvature of a specific point on a surface that clearly comes from the shape of the surface there. We can also find k_1 and k_2 in general (and therefore K) using just the chart for a surface of revolution by taking derivatives.

1.3 Gaussian curvature of a Surface of Revolution

Recall that the chart for a surface of revolution is $\mathbb{X}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$, where we assume $\alpha(v) = (f(v), g(v))$ is parametrized by arc length. The following calculations were done using Mathematica Program 1 (see Appendix). We take partial derivatives of the chart with respect to u and v, denoted $\mathbb{X}_u$ and $\mathbb{X}_v$. Then, we define:

$$
\begin{aligned}
E &= \mathbb{X}_u \cdot \mathbb{X}_u = f(v)^2 && (2)\\
F &= \mathbb{X}_u \cdot \mathbb{X}_v = 0 && (3)\\
G &= \mathbb{X}_v \cdot \mathbb{X}_v = 1 && (4)\\
N &= \frac{\mathbb{X}_u \times \mathbb{X}_v}{|\mathbb{X}_u \times \mathbb{X}_v|} = (g'(v)\cos u, g'(v)\sin u, -f'(v)) && (5)
\end{aligned}
$$

where N is the normal vector to the surface. We can use further derivatives and dot products of these quantities to find a 2×2 diagonal matrix dN_p whose entries are k_1 and k_2. Thus the determinant is the Gaussian curvature K [4, 39-47].

Remark 2. For general surfaces (not surfaces of revolution), dN_p may not be a diagonal matrix. In this case, k_1 and k_2 are its *eigenvalues*, so K is still found by computing the determinant.

To find matrix dN_p we will need the following:

$$
\begin{aligned}
l &= \mathbb{X}_{uu} \cdot N = -f(v)g'(v) && (6)\\
m &= \mathbb{X}_{uv} \cdot N = 0 && (7)\\
n &= \mathbb{X}_{vv} \cdot N = f''(v)g'(v) - f'(v)g''(v) && (8)
\end{aligned}
$$

We let $A = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ and $B = \begin{bmatrix} l & m \\ m & n \end{bmatrix}$.
Then we calculate:

$$
dN_p = A^{-1}B = \begin{bmatrix} k_1 & 0 \\ 0 & k2 \end{bmatrix} = \begin{bmatrix} -\frac{g'(v)}{f(v)} & 0 \\ 0 & f''(v)g'(v) - f'(v)g''(v) \end{bmatrix} \quad (9)
$$

Then the Gaussian curvature is the determinant, the product k_1k_2:

$$
K = \frac{g'(v)[-f''(v)g'(v) + f'(v)g''(v)]}{f'(v)}
$$

We can differentiate Equation (1) to get $f'(v)f''(v) + g'(v)g''(v) = 0$, or $f'(v)f''(v) = -g'(v)g''(v)$. We can use this to simplify the Gaussian curvature:

$$
\begin{aligned}
K &= \frac{g'(v)[-f''(v)g'(v) + f'(v)g''(v)]}{f'(v)}\\
&= \frac{1}{f(v)}\Big[-f''(v)g'(v)^2 + f'(v)(g'(v)g''(v))\Big]\\
&= \frac{1}{f(v)}\Big[-f''(v)g'(v)^2 - f'(v)(f'(v)f''(v))\Big]\\
&= -\frac{1}{f(v)}\Big[f''(v)(g'(v)^2 + f'(v)^2)\Big]\\
&= -\frac{f''(v)}{f(v)}
\end{aligned}
$$

Thus the Gaussian curvature of a surface of revolution is

$$K = -\frac{f''(v)}{f(v)} \tag{10}$$

We see from Equation (10) that the Gaussian curvature of a surface of revolution depends only on v. This is because the parameter u is just the rotation to create the surface of revolution, so the entire surface and its properties such as curvature are symmetric around the axis of rotation. The parameter v changes the position along the profile curve, which affects the curvature.

Remark 3. Given a vector $\mathbf{v}$ on a surface, we can write it in terms of the basis $\mathbb{X}_u$ and $\mathbb{X}_v$ as follows: $\mathbf{v} = u'(t)\mathbb{X}_u + v'(t)\mathbb{X}_v$. Then the squared length of this vector is

$$\begin{aligned} \|\mathbf{v}\|^2 &= \mathbf{v} \cdot \mathbf{v} \\ &= u'(t)^2\mathbb{X}_u \cdot \mathbb{X}_u + 2u'(t)v'(t)\mathbb{X}_u \cdot \mathbb{X}_v + v'(t)^2\mathbb{X}_v \cdot \mathbb{X}_v \\ &= u'(t)^2E + 2u'(t)v'(t)F + v'(t)^2G \end{aligned}$$

Example 1.2 (Gaussian curvature of a Torus)**.** We have the chart for a torus, $T(u, v) = ((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v)$. In this chart, the profile curve is *not* parametrized by arc length, so we must go back to the definitions of E, F, G, and so on to calculate the general formula for the Gaussian Curvature by finding dN_p:

$$dN_p = \begin{bmatrix} -\frac{r\cos v}{\sqrt{r^2(R+r\cos v)^2}} & 0 \\ 0 & -\frac{R+r\cos v}{\sqrt{r^2(R+r\cos v)^2}} \end{bmatrix}$$

$$K = \frac{\cos v}{rR + r^2\cos v}$$

We calculated K by plugging in $f(v) = R + r\cos v$ and $g(v) = r\sin v$ to Mathematica Program 1 (see Appendix).

Using $R = 2$ and $r = 1$ as in Figure 5, we can graph the Gaussian curvature in terms of v to see where K will be positive, zero, and negative:

$$K = \frac{\cos v}{2 + \cos v}$$

On the torus, $v = 0$ is the center height on the outside of the torus, and as angle v increases, we are going up and into the middle, then back out on the bottom (see Figure 4). Therefore, the torus has positive curvature on the outside, negative curvature on the inside, and zero curvature at the very top and bottom of the torus (where it would be tangent to a plane).

Specifically, when $v = 0$, the torus has positive curvature. When $v = \frac{\pi}{2}$, it has zero curvature, and when $v = \pi$, it has negative curvature. Figure 6 shows the principal curves at these $v-$values to connect back to Definition 1.5.

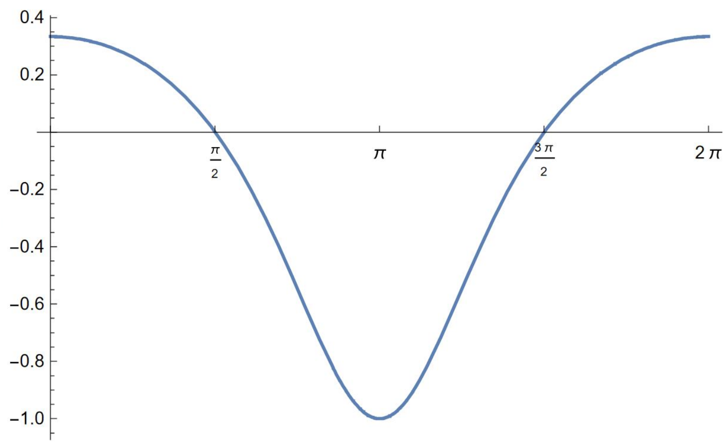

Figure 7: The Gaussian curvature K of the torus with $R = 2, r = 1$ graphed over v.

2 Geodesics

On a plane, a straight line is the shortest distance between two points. It can also be thought of as a curve satisfying the condition that the derivative of its tangent vectors is zero. We can generalize this latter condition to the concept of geodesics on surfaces. A geodesic is a curve on a surface satisfying the condition that the *covariant derivative* is zero [4, 70].

Definition 2.1 (Covariant Derivative)**.** Let $\alpha(s)$ be a curve lying on a surface S. Then the covariant derivative $\nabla(\alpha'(s))$ is the projection in the normal direction of $\alpha''(s)$ onto the tangent plane of the surface.

Definition 2.2 (Geodesic)**.** A geodesic is a curve that satisfies that $\nabla(\alpha'(s)) = 0$ for all s.

2.1 Differential Equations

A parametric equation for a particular geodesic, given a surface, a starting point, and a starting angle, can be found using a system of two differential equations that arise from setting the covariant derivative equal to zero.

Proposition 2.1 (Geodesic Differential Equations)**.** The curve $\gamma(t) = \mathbb{X}(u(t), v(t))$ is a geodesic on a surface with chart $\mathbb{X}$ if and only if it satisfies the following system of differential equations:

$$u''(t) + \Gamma^u_{uu} u'(t)^2 + 2\Gamma^u_{uv} u'(t)v'(t) + \Gamma^u_{vv} v'(t)^2 = 0 \tag{11}$$
$$v''(t) + \Gamma^v_{uu} u'(t)^2 + 2\Gamma^v_{uv} u'(t)v'(t) + \Gamma^v_{vv} v'(t)^2 = 0 \tag{12}$$

Proof. We assume that $\gamma(t) = \mathbb{X}((u(t), v(t)))$ is a geodesic on a surface with chart $\mathbb{X}$. Since $\gamma(t)$ is a geodesic, its covariant derivative must be equal to zero. We will show that this condition is equivalent to satisfying the system of differential equations. All of the calculations we do are true in both directions, satisfying the if and only if in the proposition.

We start by calculating the first and second derivatives of the geodesic [3, 262] :

$$
\begin{aligned}
\gamma'(t) =& \mathbb{X}_u u'(t) + \mathbb{X}_v v'(t) \\
\gamma''(t) =& \Big[\mathbb{X}_{uu} u'(t)u'(t) + \mathbb{X}_u u''(t) + \mathbb{X}_{uv} u'(t)v'(t) + 0\Big] \\
&+ \Big[\mathbb{X}_{uv} u'(t)v'(t) + 0 + \mathbb{X}_{vv} v'(t)v'(t) + \mathbb{X}_v v''(t)\Big] \\
\gamma''(t) =& \mathbb{X}_{uu} u'(t)^2 + \mathbb{X}_u u''(t) + 2\mathbb{X}_{uv} u'(t)v'(t) + \mathbb{X}_{vv} v'(t)^2 + \mathbb{X}_v v''(t)
\end{aligned}
$$

The three vectors $\mathbb{X}_u, \mathbb{X}_v$ and N (the normal from Equation (5)) are a basis for $\mathbb{R}^3$, so we can write the second partial derivatives $\mathbb{X}_{uu}, \mathbb{X}_{uv} = \mathbb{X}_{vu}$, and $\mathbb{X}_{vv}$ as linear combinations of that basis. We use the *Cristoffel* symbols Γ_{ij}^k as the constants in the linear combinations, where ij indicates the two partial derivatives and k indicates the first partial derivative that the symbol is associated with. The variables l, m, and n are defined as before in Equations (6) (7), and (8):

$$
\mathbb{X}_{uu} = \Gamma_{uu}^u \mathbb{X}_u + \Gamma_{uu}^v \mathbb{X}_v + lN \tag{13}
$$

$$
\mathbb{X}_{uv} = \Gamma_{uv}^u \mathbb{X}_u + \Gamma_{uv}^v \mathbb{X}_v + mN \tag{14}
$$

$$
\mathbb{X}_{vv} = \Gamma_{vv}^u \mathbb{X}_u + \Gamma_{vv}^v \mathbb{X}_v + nN \tag{15}
$$

and then use the linear combinations (13), (14), and (15) to write $\gamma''(t)$ in the basis $\{\mathbb{X}_u, \mathbb{X}_v, N\}$ as follows:

$$
\begin{aligned}
\gamma''(t) =& u'(t)^2 \Big[\Gamma_{uu}^u \mathbb{X}_u + \Gamma_{uu}^v \mathbb{X}_v + lN\Big] + \mathbb{X}_u u''(t) \\
&+ 2u'(t)v'(t)\Big[\Gamma_{uv}^u \mathbb{X}_u + \Gamma_{uv}^v \mathbb{X}_v + mN\Big] \\
&+ v'(t)^2 \Big[\Gamma_{vv}^u \mathbb{X}_u + \Gamma_{vv}^v \mathbb{X}_v + nN\Big] + \mathbb{X}_v v''(t) \\
\gamma''(t) =& \mathbb{X}_u \Big[u''(t) + \Gamma_{uu}^u u'(t)^2 + 2\Gamma_{uv}^u u'(t)v'(t) + \Gamma_{vv}^u v'(t)^2\Big] \\
&+ \mathbb{X}_v \Big[v''(t) + \Gamma_{uu}^v u'(t)^2 + 2\Gamma_{uv}^v u'(t)v'(t) + \Gamma_{vv}^v v'(t)^2\Big] \\
&+ N\big[lu'(t)^2 + mu'(t)v'(t) + nv'(t)^2\big]
\end{aligned}
$$

Since $\gamma(t)$ is a geodesic, then $\nabla(\gamma'(t)) = 0$ by definition, meaning that $\gamma''(t)$ is fully in the direction of the normal to the surface. In other words, the $\mathbb{X}_u$ and $\mathbb{X}_v$ components must equal zero. Setting the coefficients to the $\mathbb{X}_u$ and $\mathbb{X}_v$ components equal to zero gives us exactly the system of differential equations we desire. □

For a surface of revolution, the differential equations for geodesics simplify to the following [1, 232 and 255]:

$$
u'' + \frac{2ff'}{f^2} u'v' = 0 \tag{16}
$$

$$
v'' - \frac{ff'}{(f')^2 + (g')^2} u'^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2} (v')^2 = 0 \tag{17}
$$

The arguments are left out for readability; as usual f and g are $f(v)$ and $g(v)$, and u and v are $u(t)$ and $v(t)$.

The solutions u and v of this system of differential equations define the geodesic curve parametrically: the geodesic is $(u(t), v(t))$ on the $u - v$ plane, and $\gamma(t) = \mathbb{X}(u(t), v(t))$ on the surface.

Thus given a surface of revolution defined by the f and g in its chart, we can find a geodesic by choosing initial conditions for u and v in this system.

Remark 4. Since geodesics are determined by differential equations, we can apply the existence and uniqueness theorems of solutions to differential equations. Thus given any starting point and angle, there exists a unique geodesic through that point at that angle. In order for the existence and uniqueness theorems to hold, however, the terms in front of u and v in the differential equations must be continuous with continuous derivatives. Since f'' and g'' appear in Equations (16) and (17), we need f and g to have continuous *third* derivatives.

2.2 Clairaut's Relation

In the case of surfaces of revolution, a special relationship holds for geodesics, known as Clairaut's Relation, or Clairaut's Constant.

Proposition 2.2 (Clairaut's Relation)**.** Given a geodesic $\gamma(t)$, let $r(t)$ be the radius from the rotational axis of the surface to the point at t on the geodesic, and let $\theta(t)$ be the angle between $\gamma'(t)$ and $\mathbb{X}_u$ (see Figure 8). Then

$$r(t)\cos\theta(t) = c \tag{18}$$

where the constant $c = r_0 \cos\theta_0$, the starting values.

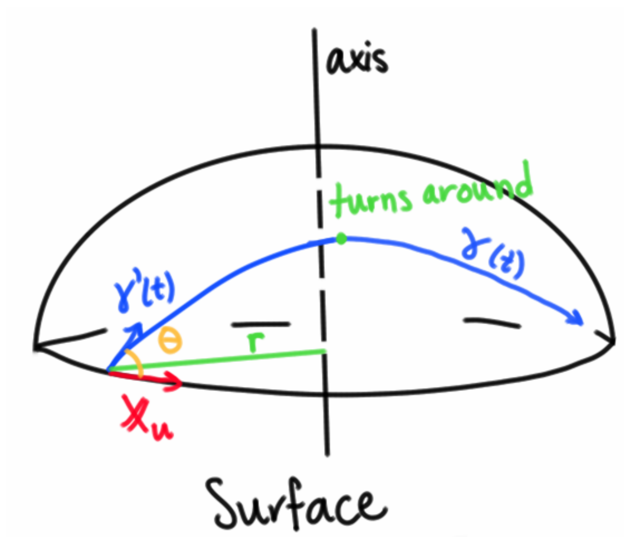

Figure 8: An example of the r and θ in Clairaut's Relation on a geodesic $\gamma(t)$.

Proof. [1, 256-257] We will use Equation (16) from the differential equations of geodesics on surfaces of revolution (where f is $f(v)$ and u, v are $u(t), v(t)$):

$$\begin{aligned} 0 &= u'' + \frac{2ff'}{f^2}u'v' \\ 0 &= f^2u'' + 2ff'u'v' \\ &= (f^2u')' \end{aligned}$$

Therefore $(f^2u') = c$ for some constant.

Now consider a geodesic γ with starting angle θ. We assume arc length parametrization of γ, so $\|\gamma'\| = 1$. Additionally, $\|\mathbb{X}_u\| = \sqrt{f(v)^2\sin^2 u + f(v)\cos^2 u} = f(v)$. Then we have

$$\begin{aligned} \gamma(t) &= \mathbb{X}(u(t), v(t)) \\ \gamma'(t) &= u'\mathbb{X}_u + v'\mathbb{X}_v \end{aligned}$$

and we use this with Equations (2) and (3) to derive

$$\begin{aligned} \cos\theta &= \frac{\gamma' \cdot \mathbb{X}_u}{\|\gamma'\|\|\mathbb{X}_u\|} \\ &= \frac{(u'\mathbb{X}_u + v'\mathbb{X}_v)\cdot\mathbb{X}_u}{f} \\ f\cos\theta &= u'\mathbb{X}_u\cdot\mathbb{X}_u + v'\mathbb{X}_u\cdot\mathbb{X}_v \\ &= u'E + v'F \\ &= f^2u' \\ &= c \end{aligned}$$

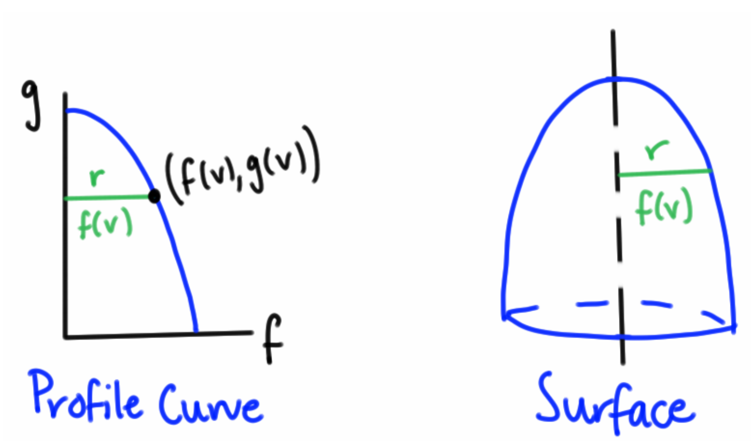

Figure 9: The way that a profile curve is turned into a surface of revolution means that $r(t) = f(v(t))$ for a surface of revolution.

Then note that $f(v)$ is equal to the radius from the rotational axis to the surface by the construction of profile curves of surfaces of revolution (see Figure 9), so we have $r\cos\theta = c$ as desired. $\square$

3 Studying Instability and Chaos

3.1 Jacobi and Riccati Equations

Now that we know how to find geodesics on a surface of revolution, we need to study whether or not they are in a chaotic system. A chaotic system is one that is comprised of unstable geodesics. To study chaos and instability, we need to look at the solutions to the Jacobi and Riccati equations, respectively.

The Jacobi and Riccati equations are as follows, where K is the Gaussian curvature along the geodesic γ:

$$J''(t) + K(\gamma(t))J(t) = 0 \tag{Jacobi}$$

$$U'(t) = -K(\gamma(t)) - U^2(t) \tag{Riccati}$$

Definition 3.1 (Strongly Unstable (J))**.** A geodesic is *strongly unstable* if the solution to the Jacobi equation with initial conditions $J(0) = 1$ and $J'(0) = 0$ satisfies $\lim_{t\to\infty} J(t) \to \infty$ <u>and</u> $J(t) > 0$ for all t.

Definition 3.2 (Strongly Chaotic)**.** A geodesic system is *strongly chaotic* if every geodesic is strongly unstable.

The Riccati equation can be derived from the Jacobi equation using the definition $U = \frac{J'}{J}$:

$$\begin{aligned} UJ &= J' \\ U'J + UJ' &= J'' = -KJ \\ U' + U\frac{J'}{J} &= -K \\ U' &= -K - U^2 \end{aligned}$$

Now that we have the connection between the Jacobi and Riccati equations, we can recast the definition of a strongly unstable geodesic using the solution to the Riccati equation.

Definition 3.3 (Strongly Unstable (U))**.** A geodesic is *strongly unstable* if the solution to the Riccati equation with initial condition $U(0) = 0$ is defined for all $t \geq 0$ and satisfies

$$\lim_{t\to\infty} \int_0^t U \to \infty.$$

Using this definition, we can see that a geodesic will be strongly unstable only if $U < 0$ for finite time periods. There must be enough regions where $U > 0$ to outweigh the negative regions.

Proposition 3.1. The two definitions of unstable, Definitions 3.1 and 3.3, are equivalent.

Proof. If we assume that $\gamma(t)$ is strongly unstable, then by Definition 3.1 it satisfies $J(0) = 1$, $J'(0) = 0$, and $\lim_{t\to\infty} J(t) \to \infty$.

Then $U(0) = \frac{J'(0)}{J(0)} = \frac{0}{1} = 0$. Furthermore, since $U = \frac{J'}{J}$, we have

$$\begin{aligned}\int_0^t U(t)dt &= \int_0^t \frac{J'(t)}{J(t)}dt \\ &= \ln J(t) - \ln J(0) \\ &= \ln J(t) - \ln 1 \\ &= \ln J(t)\end{aligned}$$

and thus

$$J = \exp \int_0^t U(t)dt.$$

Finally, since $\lim_{t\to\infty} J(t) \to \infty$, then the exponent must satisfy the limit, so $\lim_{t\to\infty} \int_0^t U \to \infty$ as desired. □

Example 3.1 (Solutions to the Riccati Equation for constant Gaussian curvature)**.** We will look at the solutions to the Jacobi and Riccati equations in the three simple cases of constant positive, zero, and constant negative Gaussian curvature. In Figures 10, 11, and 12, the graphs of $J(t)$ and $U(t)$ are shown, where $J(t)$ is solved for with the initial conditions in Definition 3.1 using Mathematica Program 5 (see Appendix). Based on the Riccati equation, we can see:

1. $K = 1$: $U' = -1 - U^2$, so U' is always negative (see Figure 10).
2. $K = 0$: $U' = -U^2$, so U' is always nonpositive. In the case with the initial condition $U(0) = 0$, then $U' = 0$ and $U(t) = 0$ for all t (see Figure 11).
3. $K = -1$: $U' = 1 - U^2$, so U' is positive if $U^2 < 1$ (see Figure 12).

From these cases of constant curvature, we can see that only negative Gaussian curvature can produce the *possibility* of a unstable geodesic, but does not guarantee it. If $K > 0$, U' is always negative, which means that U will decrease, eventually reach negative infinity, and become undefined. If $K = 0$, then U stays constant at 0, so its integral will be 0. Therefore neither of these nonnegative cases can produce a strongly unstable geodesic.

Most surfaces, however, do not have constant curvature as in these examples, so U' will be negative in some parts and positive in others. What we need for our surface to have the possibility of strongly unstable geodesics is for there to be *enough* negative curvature to outweigh the parts of the surface with nonnegative curvature. It is also important for the positive curvature to be spread out over the surface so that its effect is weaker. For example, in Figure 13, $K = -1$, so $-\sqrt{-K} = -1$, and this line is plotted. We see that U is starting below $-\sqrt{-K}$ and goes to negative infinity when $J(t) = 0$, at $t = 1.5$. When $J(t) = 0$, it breaks Definitions 3.1 that $J > 0$. At the same time, U goes to negative infinity, making it impossible to satisfy Definition 3.3. If an area of positive curvature made U drop below $-\sqrt{-K}$, then no amount of negative curvature would be able to bring U back up to satisfy being strongly unstable.

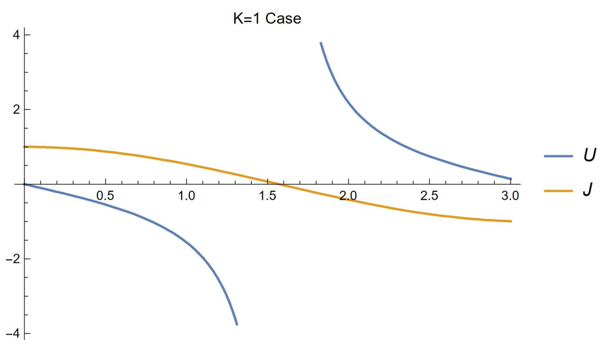

Figure 10: Graphs of U and J for $K = 1$.

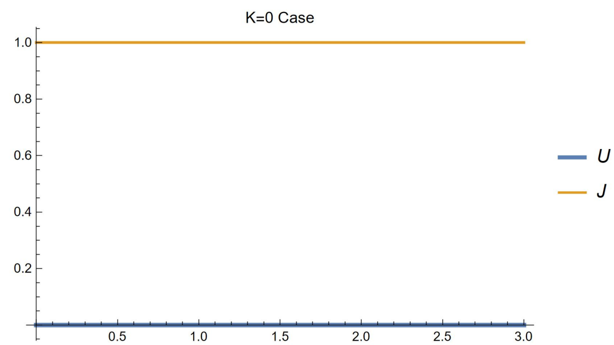

Figure 11: Graphs of U and J for $K = 0$.

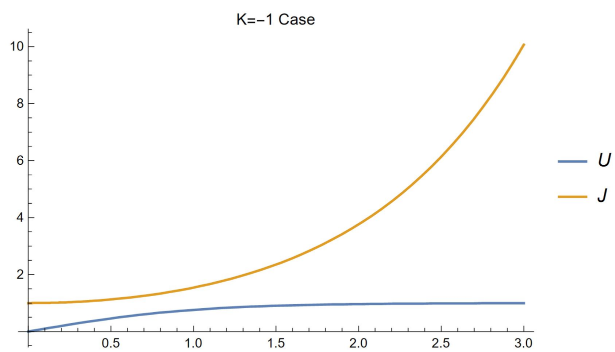

Figure 12: Graphs of U and J for $K = -1$.

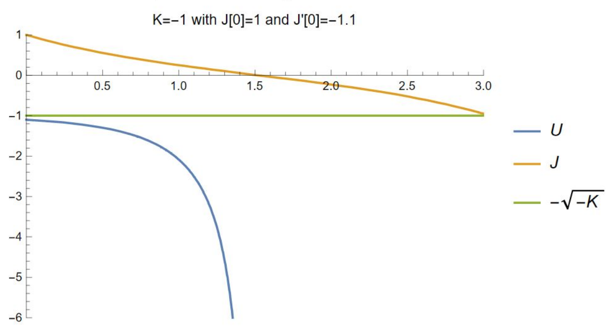

Figure 13: Graphs of U, J, and K for $K = -1$ with initial conditions $J(0) = 1$, $J'(0) = -1.1$.

For more complex curvature, we can use a clever graph to see the behavior of U: if we plot $+\sqrt{-K}, -\sqrt{-K}$, for $K < 0$, and U, then U will fall when it is above or below both $\sqrt{-K}$ curves and rise in the middle. If K is positive, then U always falls.

Example 3.2 (Solution to the Riccati Equation for a Torus)**.** Here is the graph of U, with $+\sqrt{-K}$ and $-\sqrt{-K}$ shown, for the profile curve of a torus with $R = 2$ and $r = 1$. In the regions $[0, \frac{\pi}{2}]$ and $[\frac{3\pi}{2}, 2\pi]$ the Gaussian curvature is positive (see Figure 7) so the $\sqrt{-K}$ curves do not exist.

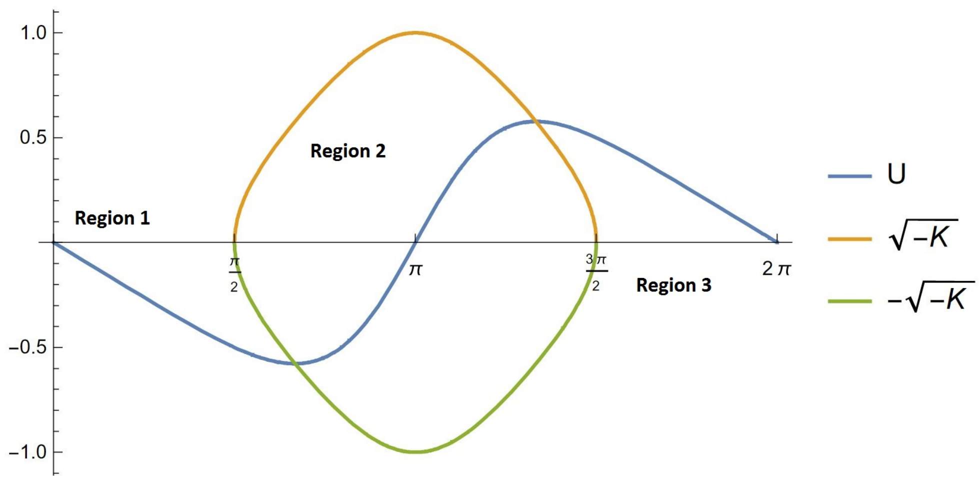

Figure 14: Graph of the solution to the Riccati Equation for a torus.

Let's examine what is happening in each region of this graph:

1. $[0, \frac{\pi}{2}]$: K is positive, so U falls.
2. $[\frac{\pi}{2}, \frac{3\pi}{2}]$: $K < 0$, and initially U is still below both $\sqrt{-K}$ curves and falling. However, the $-\sqrt{-K}$ curve falls steeper, so it "catches" the U curve, which rises until it gets higher than $+\sqrt{-K}$.
3. $[\frac{3\pi}{2}, 2\pi]$: K is positive again, so U falls.

Since we are on the profile curve of the torus, a circle, at 2π we return to the beginning. We can see that at 2π, U returns to its starting point of zero, so this U graph will just cycle around again. We see that U is symmetric, so the positive and negative areas cancel to zero. Thus the integral of U will be zero, and the geodesic that is the profile curve of the torus is not strongly unstable.

4 Creating a Surface of Revolution

4.1 Finding a Curve With Given Space Curvature

In creating a surface of revolution, we are really just designing a profile curve. To do so, it is useful to be able to find a space curve that has a chosen space curvature function.

Proposition 4.1. Given a space curvature as a function of time $\kappa(t)$, then the space curve $\alpha(t) = (x(t), y(t))$ is parametrized by arc length and its space curvature is $\kappa(t)$, where $x(t)$ and $y(t)$ are the solutions to the following system of differential equations [3, 111-113]:

$$x'(t) = \cos(\theta(t)) \tag{19}$$
$$y'(t) = \sin(\theta(t)) \tag{20}$$
$$\theta'(t) = \kappa(t) \tag{21}$$

Proof. We can verify that $\alpha(t) = (x(t), y(t))$ has space curvature $\kappa(t)$ by calculating it directly:

$$\begin{aligned} x''(t) &= -\sin(\theta(t))\theta'(t) = -\kappa(t)\sin(\theta(t)) \\ y''(t) &= \cos(\theta(t))\theta'(t) = \kappa(t)\cos(\theta(t)) \end{aligned}$$

and the space curvature is

$$\begin{aligned} \|\alpha''(t)\| &= \sqrt{x''(t)^2 + y''(t)^2} \\ &= \sqrt{(-\kappa(t)\sin(\theta(t)))^2 + (\kappa(t)\cos(\theta(t)))^2} \\ &= \sqrt{\kappa(t)^2[\sin^2(\theta(t)) + \cos^2(\theta(t))]} \\ &= \kappa(t). \end{aligned}$$

Furthermore, we can easily see that the curve resulting from these equations will be parametrized by arc length, meaning $\|\alpha'(t)\| = 1$ for all t:

$$\begin{aligned} \|\alpha'(t)\| &= \sqrt{x'(t)^2 + y'(t)^2} \\ &= \sqrt{(\cos(\theta(t)))^2 + (\sin(\theta(t)))^2} \\ &= 1. \end{aligned}$$

□

Creating a Profile Curve and a Surface of Revolution We will use this system of equations to create a profile curve for a surface of revolution by setting $f(t) = x(t)$ and $g(t) = y(t)$. Then the profile curve will be $(f(v), g(v))$, and we can turn it into a surface of revolution.

4.2 Our Desired Space Curvature Function

We want a surface that has a tube in the center like the torus, but the top is shaped differently because the top of the torus has too much positive curvature. Circles have constant space curvature, so our idea is to have a profile curve that has a constant space curvature to start

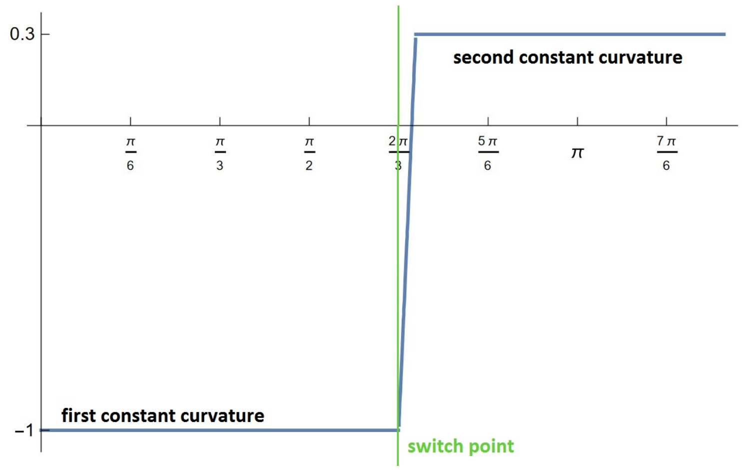

Figure 15: The piecewise curve specified in Equation (22) with two constant curvatures connected by a straight line.

(a circle like the inside of the torus), then changes to a constant space curvature of opposite sign to avoid the positive curvature on the surface (like the torus has). We will connect these two constants with a line over a switch distance of 0.1.

Let κ_1 and κ_2 be the two constant space curvatures, and let w be the switch point value of t. Then the general formula for this piecewise space curvature is:

$$\kappa_g(t) = \begin{cases} \kappa_1 & 0 < s \leq w \\ 10(\kappa_2 - \kappa_1)(s - w) + \kappa_1 & w < s \leq w + 0.1 \\ \kappa_2 & w + 0.1 < s \leq 2w \end{cases}$$

Changing the values of κ_1, κ_2, and the switch point create different profile curves (and therefore surfaces of revolution) after going through the system of differential equations (19), (20), and (21). We want one that flattens out on the outside. We end up choosing constants -1 and 0.3 for the space curvature, with a switch point of $\frac{2\pi}{3}$. The specific space curvature function we plug into the differential equations from above is:

$$\kappa(t) = \begin{cases} -1 & 0 < s \leq \frac{2\pi}{3} \\ 13(s - \frac{2\pi}{3}) - 1 & \frac{2\pi}{3} < s \leq \frac{2\pi}{3} + 0.1 \\ 0.3 & \frac{2\pi}{3} + 0.1 < s \leq \frac{4\pi}{3} \end{cases} \tag{22}$$

This piecewise curve is plotted in blue in Figure 16.

Improving the Space Curvature Function As noted in Remark 4, the f and g we use in the profile curve should have continuous third derivatives. Note that by Equations (19) and (21), where $f(t) = x(t)$, then the third derivative of f is as follows:

$$f'(t) = \cos\theta(t) \tag{23}$$

$$f''(t) = -\sin\theta(t)\kappa(t) \tag{24}$$

$$f'''(t) = -\cos\theta(t)\kappa^2(t) - \sin\theta(t)\kappa'(t) \tag{25}$$

and thus the third derivative of f (and similarly of g) has $\kappa'(t)$ in it. Then if $\kappa(t)$ is piecewise-defined with three straight lines as in Equation (22), $\kappa'(t)$ will not exist, so $f'''(t)$ and $g'''(t)$ will not exist. Hyperbolic tangent, on the other hand, has a continuous derivative, so it is smooth enough to use for $\kappa(t)$. Thus we use it to make an improved space curvature function.

Hyperbolic tangent moves from output values near constant -1 to $+1$, and we want our function to go from -1 to 0.3. We scale the height of the function by multiplying it by $\frac{1.3}{2}$. This gives us a function with the correct height, but it starts at the wrong value. Thus we adjust by adding in the difference between -1 and the value at 0. Lastly we subtract $(\frac{2\pi}{3} - 0.05)$ from t inside the hyberbolic tangent to horizontally shift the switch point from 0 to approximately $\frac{2\pi}{3}$, and divide by 0.1 to shorten the time it takes to switch. Thus the improved space curvature function is:

$$\kappa_{tanh}(t) = \frac{1.3}{2}\tanh\left(\frac{x - \frac{2\pi}{3} + 0.05}{0.1}\right) + \left[-1 - \frac{1.3}{2}\tanh\left(\frac{-\frac{2\pi}{3} + 0.05}{0.1}\right)\right] \tag{26}$$

This curve is plotted in orange in Figure 16 to compare to the original piecewise function.

We tried out different space curvature functions before deciding on these values using Mathematica Program 2 (see Appendix).

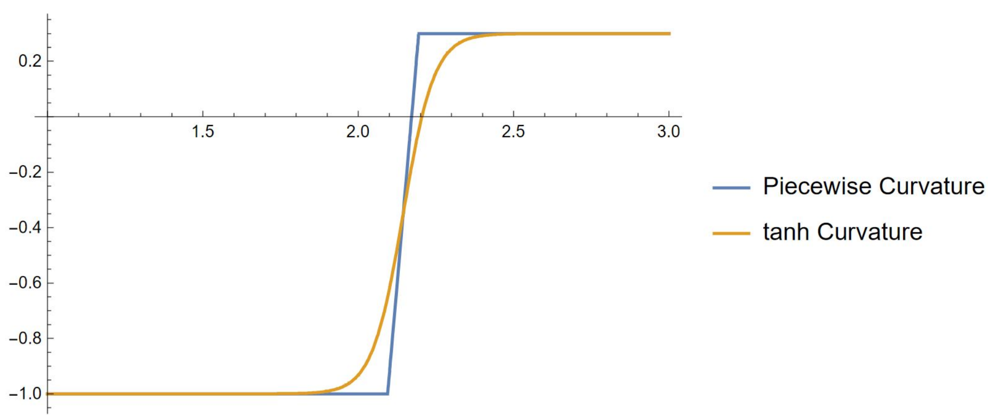

Figure 16: The graphs of the original piecewise curvature connecting two constants with a line compared to the improved hyperbolic tangent curvature.

For the remainder of the paper we will be using the hyperbolic tangent space curvature (26) and its resulting profile curve and surface of revolution.

Our Initial Surface Using the hyperbolic tangent function as the space curvature, we get the profile curve in Figure 17 and the tube in Figure 18. Note that $v = 0$ is on the inside of the tube, so the values of v will decrease from the outside to the inside of the tube. These graphs are defined in Mathematica Program 3a (see Appendix).

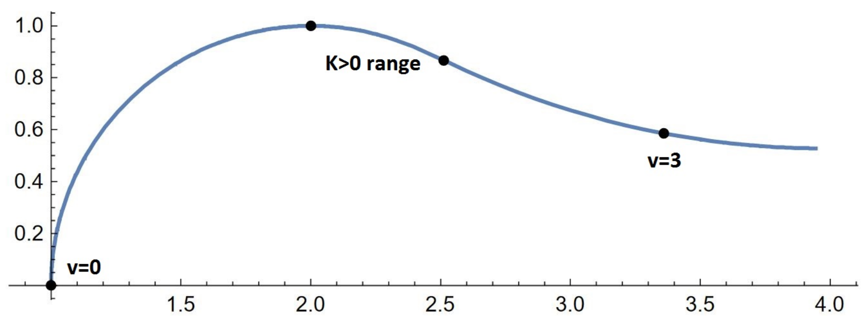

Figure 17: The profile curve with the hyperbolic tangent function as its space curvature. Shown are the points $v = 0$, $v = 3$, and the boundary points for where the surface goes into positive curvature.

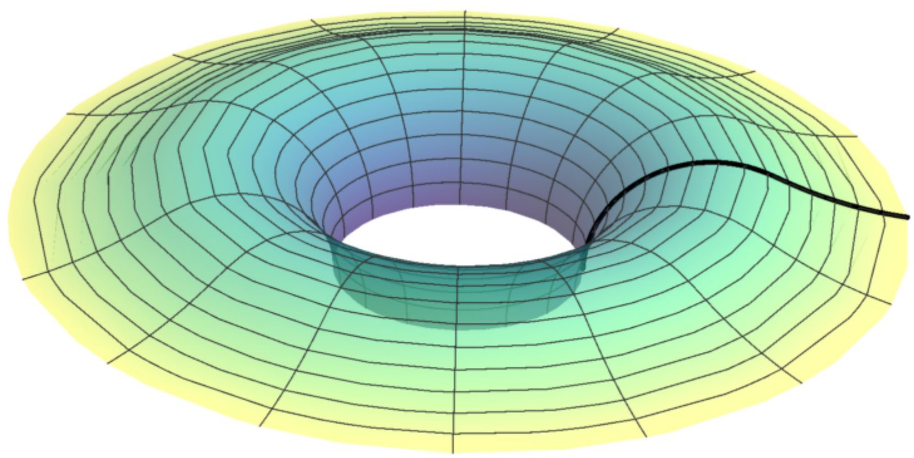

Figure 18: The tube created from the hyperbolic tangent profile curve, with the profile curve outlined in black.

5 Analyzing Our Surface

5.1 Finding the Positive Curvature Region

Remark 5. Recall from Equation (10) that $K(v) = -\frac{f''(v)}{f(v)}$. We can see how this is connected to the space curvature $\kappa(t)$ we used to create the profile curve using (24):

$$K(v) = -\frac{f''(v)}{f(v)} = \frac{\sin\theta(t)\kappa(t)}{f(v)}$$

Additionally, to find the Gaussian curvature *along* a geodesic $\gamma(t) = \mathbb{X}(u(t), v(t))$ we simply calculate $K(v(t)) = -\frac{f''(v(t))}{f(v(t))}$.

As discussed in Section 3.1, negative curvature produces the possibility of unstable geodesics, and positive curvature is bad if there is too much of it. Thus it would be helpful to know where our tube has positive Gaussian curvature. We numerically solve $K(v) = 0$, using Mathematica Program 3b (see Appendix), to find that the region where $K > 0$ is $1.5708 < v < 2.10462$.

5.2 Initial Conditions for Solving for Geodesics

The geodesic differential equations (16) and (17) need initial conditions $u(0) = u_0, v(0) = v_0, u'(0) = u'_0$, and $v'(0) = v'_0$. These initial conditions give the location and direction of a vector $\mathbf{z} = \langle u'_0, v'_0 \rangle$ in the $u - v$ plane. The geodesic that we will study, however, is on the surface. We need to understand how direction vector $\mathbf{z}$ is related to $\mathbf{w} = d\mathbb{X}\mathbf{z}$, the corresponding initial direction vector for the geodesic on the surface, so we can effectively choose the initial conditions. The position (u_0, v_0) on the $u-v$ plane simply goes to $\mathbb{X}(u_0, v_0)$ on the surface, so we need only study u'_0 and v'_0.

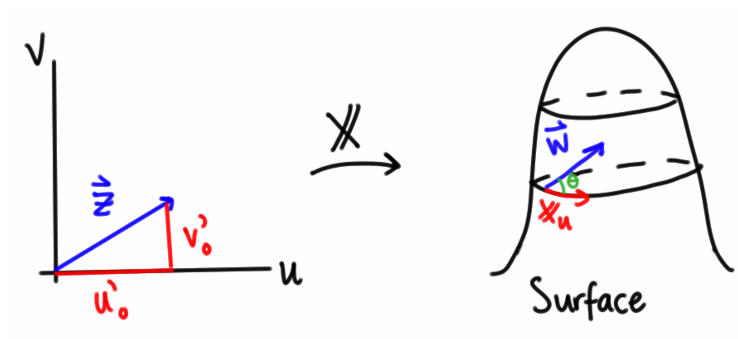

Figure 19: The vectors $\mathbf{z}$ and $\mathbf{w}$ and angle θ.

Proposition 5.1. Let $\mathbf{z} = \langle u'_0, v'_0 \rangle$ be a vector in the $u-v$ plane. If $u'_0 = \frac{\cos\theta}{f(v)}$ and $v'_0 = \sin\theta$, then $\mathbf{w} = d\mathbb{X}\mathbf{z}$ has unit length and makes an angle of θ with the $\mathbb{X}_u$ vector on the surface.

Proof. First we compute: $\mathbf{w} = d\mathbb{X}\mathbf{z} = \begin{bmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g(v) \end{bmatrix} \begin{bmatrix} u'_0 \\ v'_0 \end{bmatrix}$, so we have

$$\mathbf{w} = \begin{bmatrix} -f(v)u'_0 \sin u + f'(v)v'_0 \cos u \\ f(v)u'_0 \cos u + f'(v)v'_0 \sin u \\ g'(v)v'_0 \end{bmatrix} \tag{27}$$

Then $\|\mathbf{w}\|^2 = f(v)^2(u'_0)^2 + (v'_0)^2$ (using the fact that f is parametrized by arc length). Now we calculate:

$$\begin{aligned} \|\mathbf{w}\|^2 &= f(v)^2(u'_0)^2 + (v'_0)^2 \\ &= f(v)^2 \left(\frac{\cos\theta}{f(v)}\right)^2 + (\sin\theta)^2 \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

as desired.

Now we need to verify that the angle between $\mathbf{w}$ and $\mathbb{X}_u$ is θ. Let this angle be ϕ. Recall that $\|\mathbb{X}_u\| = \sqrt{f(v)^2 \sin^2 u + f(v)\cos^2 u} = f(v)$. Then

$$\cos\phi = \frac{\mathbf{w}\cdot\mathbb{X}_u}{\|\mathbf{w}\|\|\mathbb{X}_u\|} = \frac{f(v)^2 u_0'}{1\cdot f(v)} = f(v)u_0' = f(v)\frac{\cos\theta}{f(v)} = \cos\theta$$

□

5.3 Calculating Geodesics

We use Mathematica Program 3c (see Appendix) to solve the differential equations (16) and (17) with initial conditions

$$u(0) = 0 \qquad v(0) = 3$$
$$u'(0) = \frac{\cos(-\theta)}{f(3)} \qquad v'(0) = \sin(-\theta)$$

for geodesics starting on the outside of the tube (at $v = 3$, in the negative curvature region) pointing with direction θ towards the inside of the tube. The choice of 3 for $v(0)$ is somewhat arbitrary; it simply must be in the part of the surface that has negative curvature. If we use $\theta = \pi/2$ we will get the geodesic that is exactly the profile curve. Then we plug in $v(t)$ to the curvature formula to get K along the geodesic, and use this K in the Riccati equation to solve for U.

We can use a corollary of Clairaut's Relation to get more information about how to choose the starting angle of the geodesic.

Corollary 5.1. A geodesic that turns around at $v = v_{turn}$ has starting angle

$$\theta_0 = \cos^{-1}\left(\frac{f(v_{turn})}{f(3)}\right) \tag{28}$$

Proof. Recall from the proof of Proposition 2.2 that $r(t) = f(v(t))$.

Then we can rewrite Clairaut's Relation in terms of f instead of r:

$$\begin{aligned} c &= r(t)\cos(\theta(t)) \\ &= f(v(t))\cos(\theta(t)) \end{aligned}$$

Additionally,

$$\begin{aligned} c &= r_0\cos\theta_0 \\ &= f(v_0)\cos\theta_0 \\ &= f(3)\cos\theta_0 \end{aligned}$$

since we have chosen $v_0 = 3$.

Thus we have

$$f(3)\cos\theta_0 = f(v(t))\cos(\theta(t)) \tag{29}$$

A geodesic "turning around" on the surface means that it makes an angle of $\theta_{turn} = 0$ with $\mathbb{X}_u$, so $\cos\theta_{turn} = 1$. We plug $(v_{turn}, \theta_{turn})$ into Equation 29:

$$\begin{aligned} f(3)\cos\theta_0 &= f(v_{turn})\cos\theta_{turn} \\ &= f(v_{turn}) \\ \Rightarrow \cos\theta_0 &= \frac{f(v_{turn})}{f(3)} \end{aligned}$$

Therefore $\theta_0 = \cos^{-1}\left(\frac{f(v_{turn})}{f(3)}\right)$ as desired. □

Angles Corresponding to Positive Gaussian Curvature Using Equation (28), we can calculate the critical starting angles that divide geodesics based on when they turn around in relation to the Gaussian curvature of the surface.

Corollary 5.2. Let θ_1 and θ_2 be the starting angles of the geodesics that turn around at the boundaries of the positive curvature region of the surface, and let θ_3 be the starting angle of the geodesic that does not turn around but circles around the inner "equator" of the tube (at $v = 0$). Then, for our surface defined by the space curvature in Equation (26), we have

$$\begin{aligned} \theta_1 &= 0.726157 \\ \theta_2 &= 0.93305 \\ \theta_3 &= 1.26852 \end{aligned}$$

Proof. We calculate these angles using Equation (28) in Mathematica Program 4 (see Appendix). In Section 5.1, we calculated that the surface has positive Gaussian curvature when $v \in [1.5708, 2.10462]$. Thus we use those $v-$values to get θ_1 and θ_2. Then to find θ_3, we plug in $v = 0$. □

Geodesics with starting angles smaller than θ_1 will only be in positive curvature regions of the surface, and those with starting angles larger than θ_2 will make it all the way through the positive curvature region back into negative curvature. Thus geodesics with starting angles in the range $[\theta_1, \theta_2]$ are in the most danger of failing to satisfy the conditions of being strongly unstable because they spend the most time in the positive curvature region. As for θ_3, this is the starting angle of the first geodesic that does not turn around, specifically the geodesic that gets "stuck" at the equator. Geodesics with starting angles larger than θ_3 will *not* turn around and return to $v = 3$; they will go inside the tube. If the tube were part of the whole surface described in the introduction, the geodesic would then go into the inner sphere.

Remark 6. In looking at these geodesics and the graphs of their solutions to the Riccati equation, we are only looking at a short time period of the geodesic (either until it returns to $v = 3$ or when it reaches the end of the tube at $v = 0$). Definition 3.3 of strongly unstable geodesics uses the integral of U until positive infinity, as opposed to over a finite length of the geodesic.

In the following section, we will be categorizing geodesics as strongly unstable based on the portion we can see on the tube. When the geodesic leaves our tube, however, it will go onto the surface we imagine the tube to be a part of. Thus, geodesics which appear strongly unstable on the tube are only *potentially* strongly unstable on the surface, where we can calculate the integral of U as t goes to infinity. That surface will need to be designed such that the geodesics which are potentially strongly unstable on the tube can fully satisfy the condition of being strongly unstable on the entire surface. The proposed hydrant surface (see Figure 2) has negative curvature everywhere, which would ensure that solutions to the Riccati equation that start above $-\sqrt{-K}$ would stay above it and be strongly unstable.

5.4 Analyzing Graphs

Once we have solved for a geodesic $\gamma(t) = \mathbb{X}(u(t), v(t))$ and for K and U, we look at three graphs to study the geodesic. All of the graphs are plotted for the length of time it takes the geodesic to return to $v_0 = 3$, except for the last one which enters the tube and does not return. These graphs were created using Mathematica Program 3c (see Appendix).

1. $\gamma(t)$ on the tube with the positive curvature region outlined: this allows us to see if the geodesic goes into the positive curvature region. If it does, we can see how long it spends there before coming back or going into the tube.

2. $K(v(t))$, the Gaussian curvature along the geodesic: we look to see if K becomes positive, and if so, how positive and for how long.

3. $U, \pm\sqrt{-K}$: we want to see whether U ends up either between or above both $\pm\sqrt{-K}$ lines, or else falls below them both. If U ends up above the $-\sqrt{-K}$ line when it returns to $v_0 = 3$, then $\gamma(t)$ is strongly unstable (see Remark 6). *(In these graphs, U is the green line.)*

Here we show these three graphs for seven geodesics with the following starting angles:

1. $0 < \frac{\pi}{6} \approx 0.524 < \theta_1$ (turns around before $K > 0$ region)

2. $\theta_1 < \frac{\pi}{4} \approx 0.785 < \theta_2$ (turns around in $K > 0$ region)

3. $\theta_1 < 0.225\pi \approx 0.801 < \theta_2$ (turns around in $K > 0$ region)

4. $\theta_2 < 0.31\pi \approx 0.974 < \theta_3$ (turns around after $K > 0$ region)

5. $\theta_2 < \frac{\pi}{3} \approx 1.047 < \theta_3$ (turns around after $K > 0$ region)

6. $\theta_2 < 0.4\pi \approx 1.257 < \theta_3$ (makes it into the tube but still turns around)

7. $\theta_3 < 0.404\pi \approx 1.269 < \frac{\pi}{2}$ (never turns around; ends up in the tube)

1. This geodesic is only in the negative curvature region of the surface. We can see that U is between the $\pm\sqrt{-K}$ curves and so it rises for the time it takes to return to $v = 3$. It is strongly unstable.

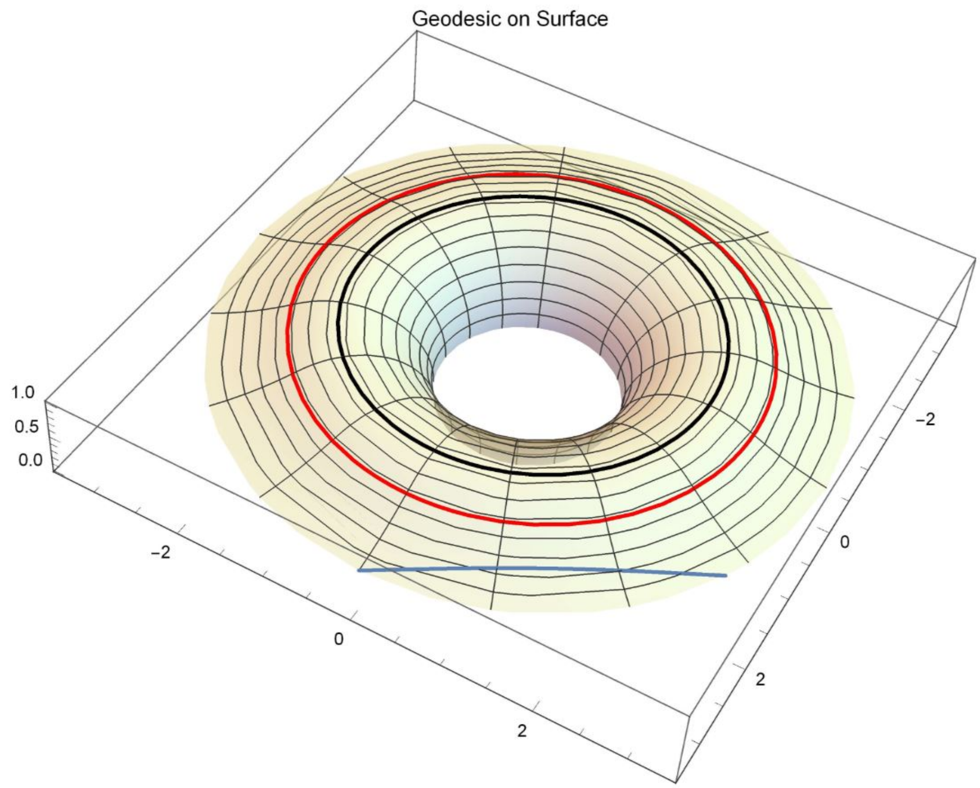

Figure 20: Geodesic with starting angle $\frac{\pi}{6}$

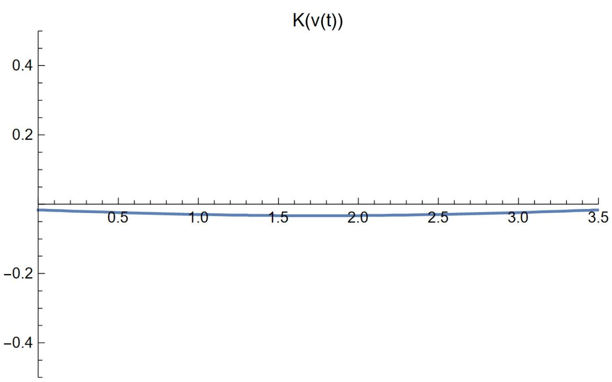

Figure 21: K along geodesic with starting angle $\frac{\pi}{6}$

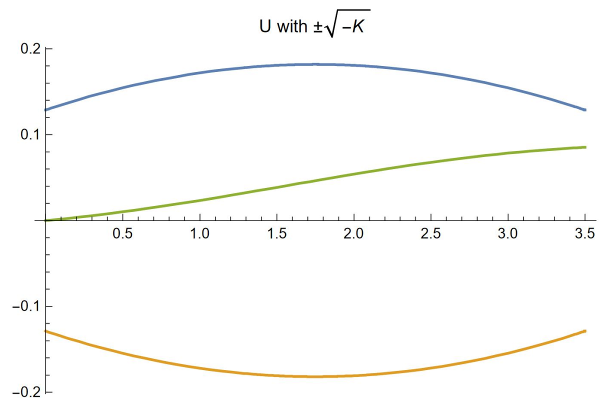

Figure 22: U along geodesic with starting angle $\frac{\pi}{6}$

2. This geodesic starts in negative curvature, and then goes into the $K > 0$ region briefly. During this time, U falls below zero, but not below $-\sqrt{-K}$, so when it gets back into $K < 0$ it rises again. It is strongly unstable.

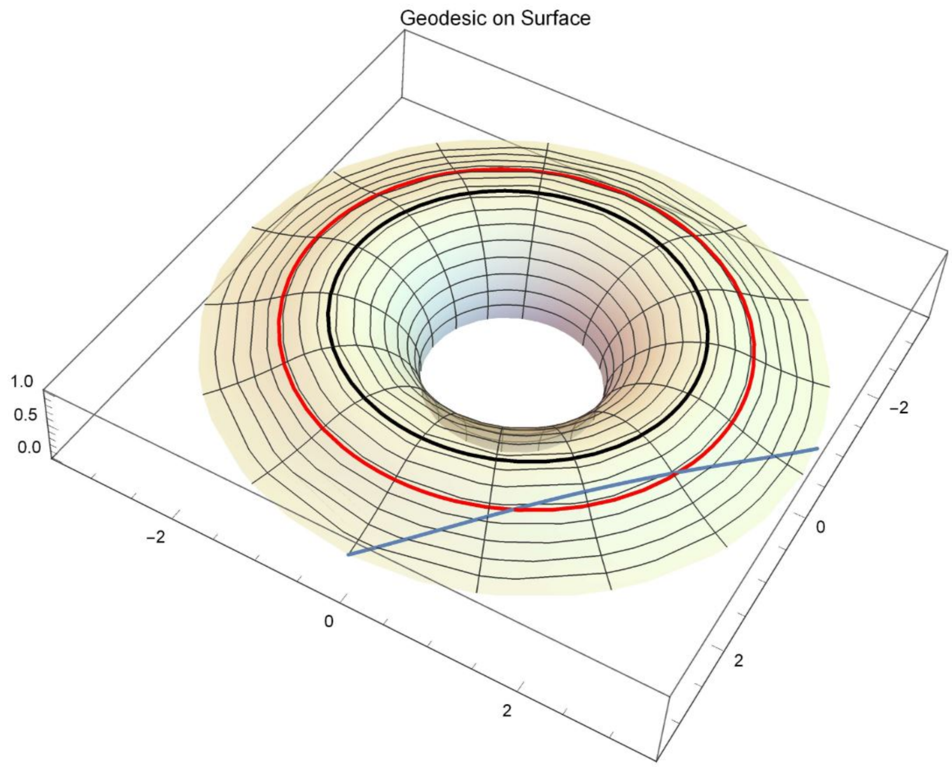

Figure 23: Geodesic with starting angle $\frac{\pi}{4}$

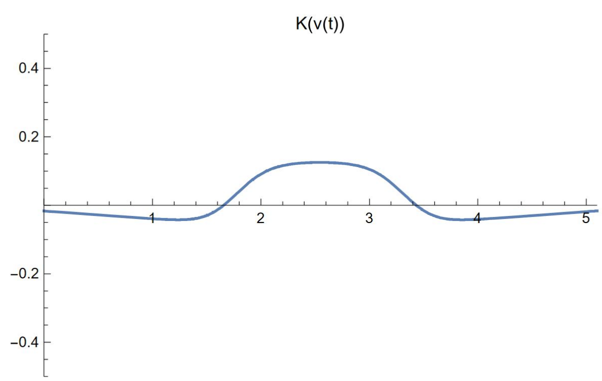

Figure 24: K along geodesic with starting angle $\frac{\pi}{4}$

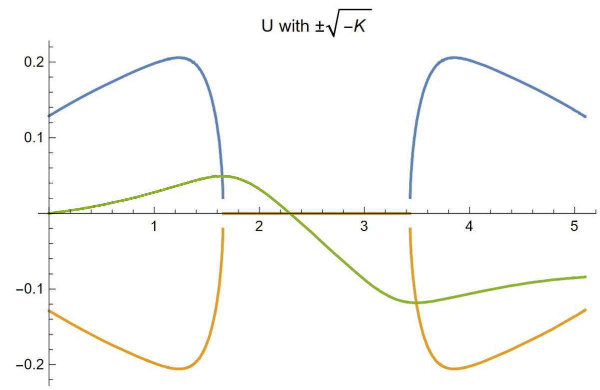

Figure 25: U along geodesic with starting angle $\frac{\pi}{4}$

3. The solution to the Riccati equation begins the same way for this geodesic as for the previous, but it stays in positive curvature for longer. When it goes back into negative curvature, it is still above $-\sqrt{-K}$, but the curvature is flattening out, and it falls below before it returns to $v_0 = 3$. Since U has fallen below both $\sqrt{-K}$ curves, it will fall to negative infinity like the case in Figure 13. It is *not* strongly unstable.

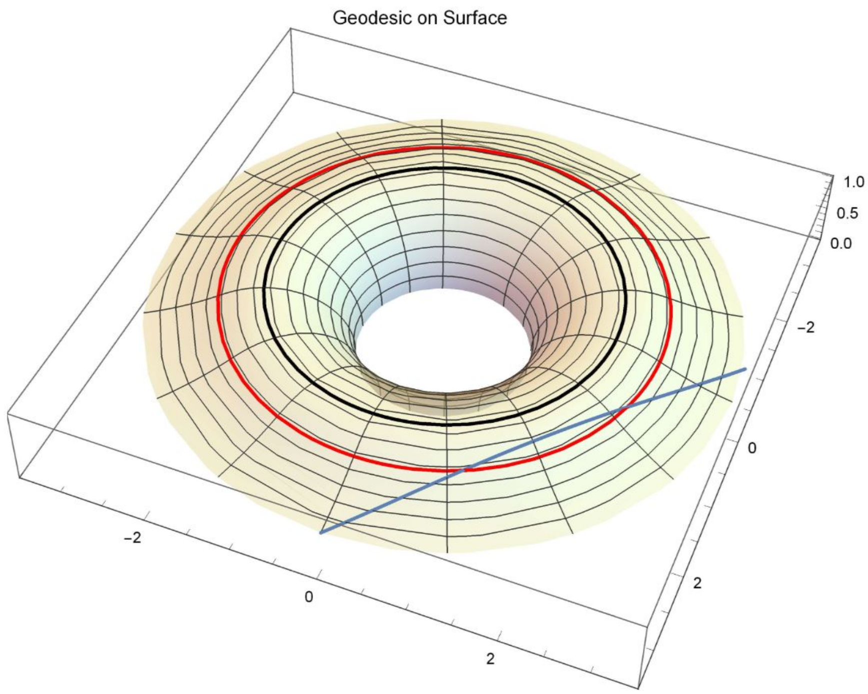

Figure 26: Geodesic with starting angle 0.255π

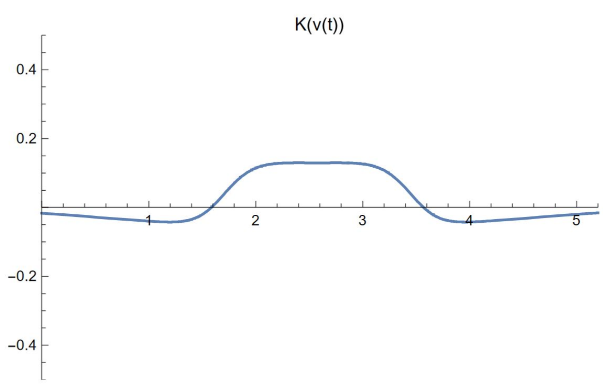

Figure 27: K along geodesic with starting angle 0.255π

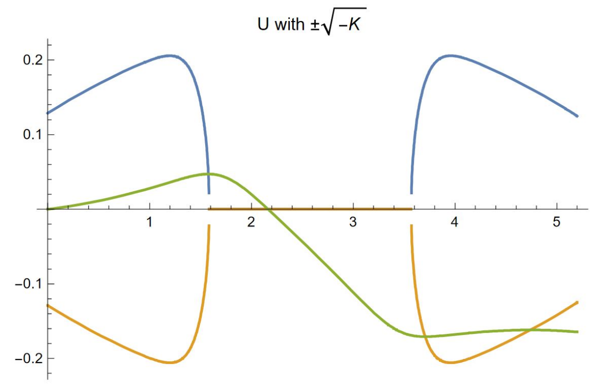

Figure 28: U along geodesic with starting angle 0.255π

4. This geodesic goes all the way through the $K > 0$ region before turning around, so it has a middle section of negative curvature. Thus U alternates between rising and falling as the geodesic alternates between positive and negative curvature. In the end, U is above $-\sqrt{-K}$, so the geodesic is strongly unstable.

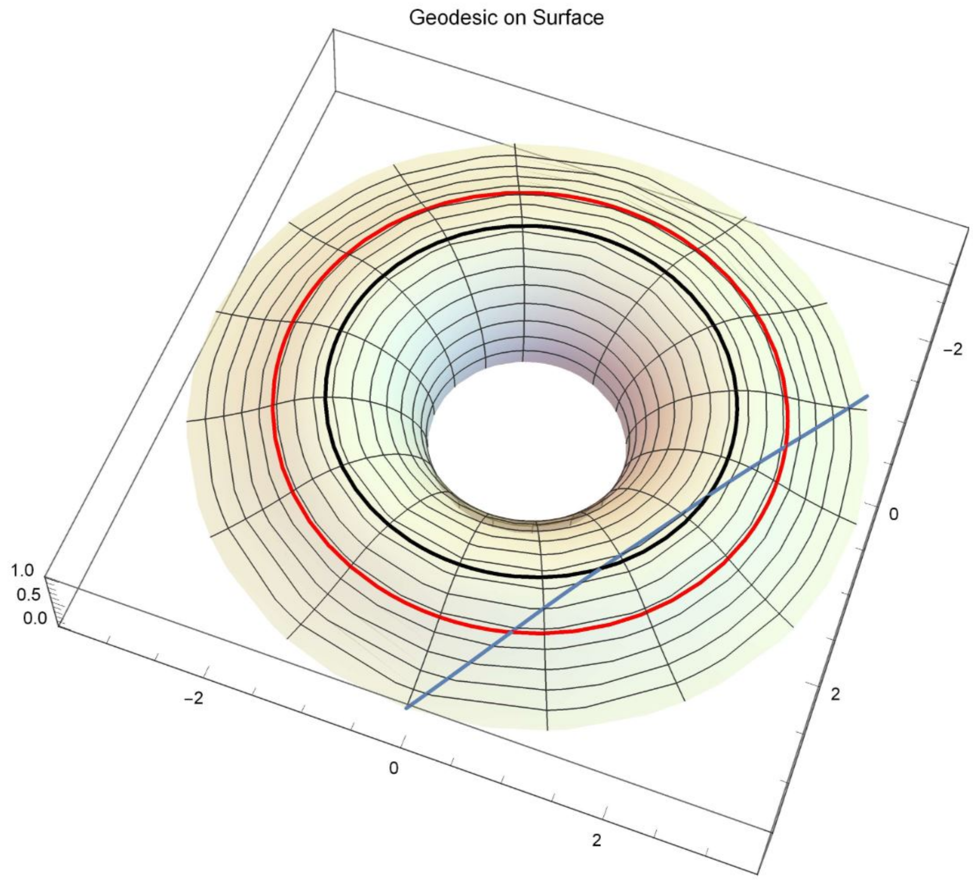

Figure 29: Geodesic with starting angle 0.31π

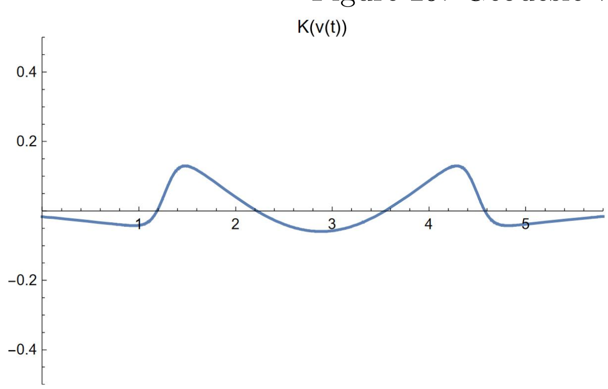

Figure 30: K along geodesic with starting angle 0.31π

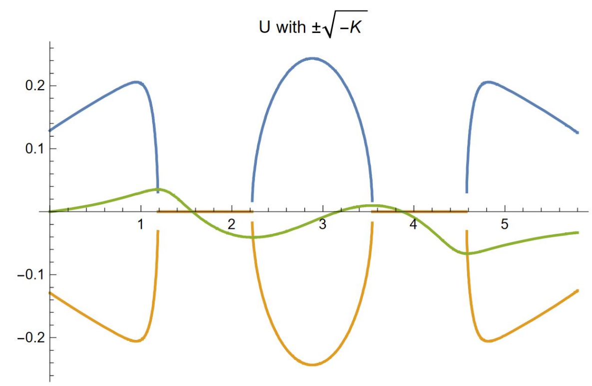

Figure 31: U along geodesic with starting angle 0.31π

5. This geodesic has the same trajectory as the previous one, but spends longer in negative curvature in the middle section. This allows U to rise even farther, so it ends up above $+\sqrt{-K}$. It is strongly unstable.

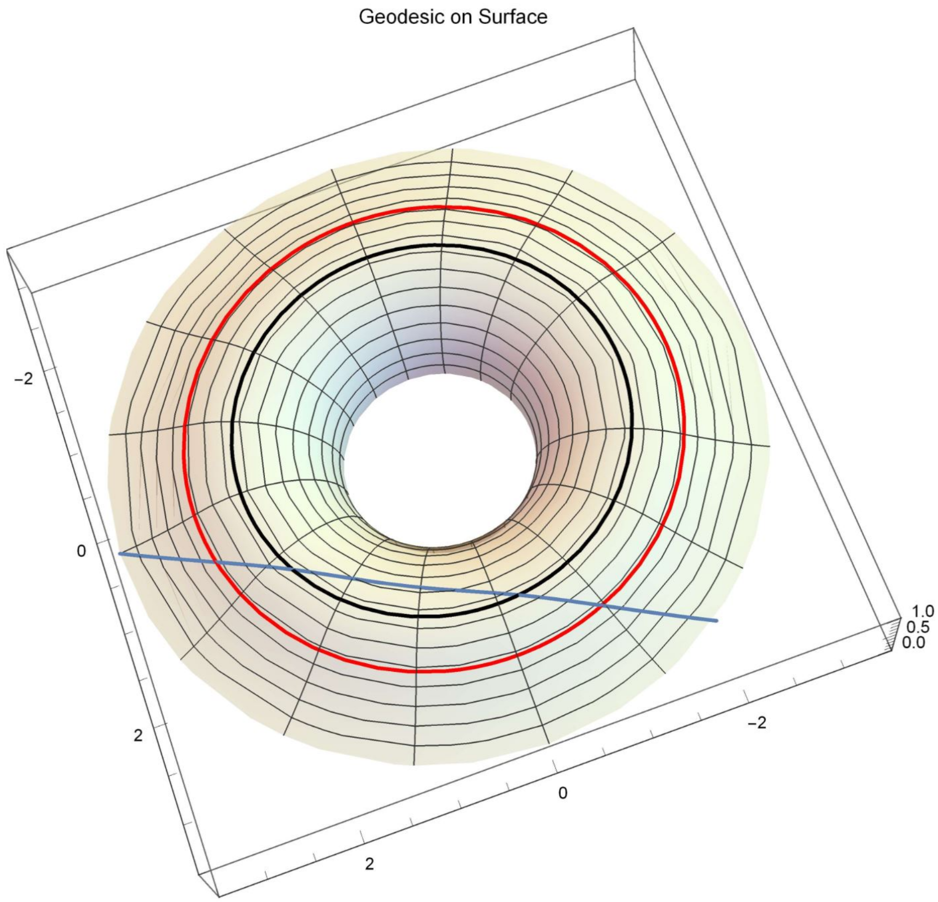

Figure 32: Geodesic with starting angle $\frac{\pi}{3}$

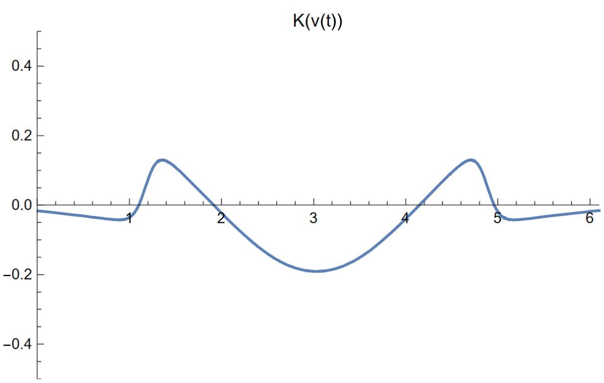

Figure 33: K along geodesic with starting angle $\frac{\pi}{3}$

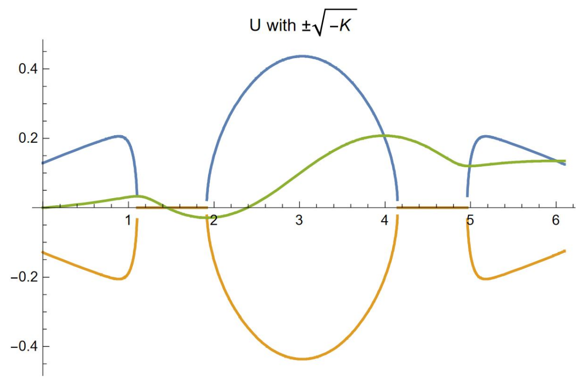

Figure 34: U along geodesic with starting angle $\frac{\pi}{3}$

6. This geodesic quickly passes through the $K > 0$ region and then spends a long time in the $K < 0$ region, actually entering the tube which has more strongly negative curvature. Thus U rises above $+\sqrt{-K}$ while the geodesic is inside the tube, and it stays above. It is strongly unstable.

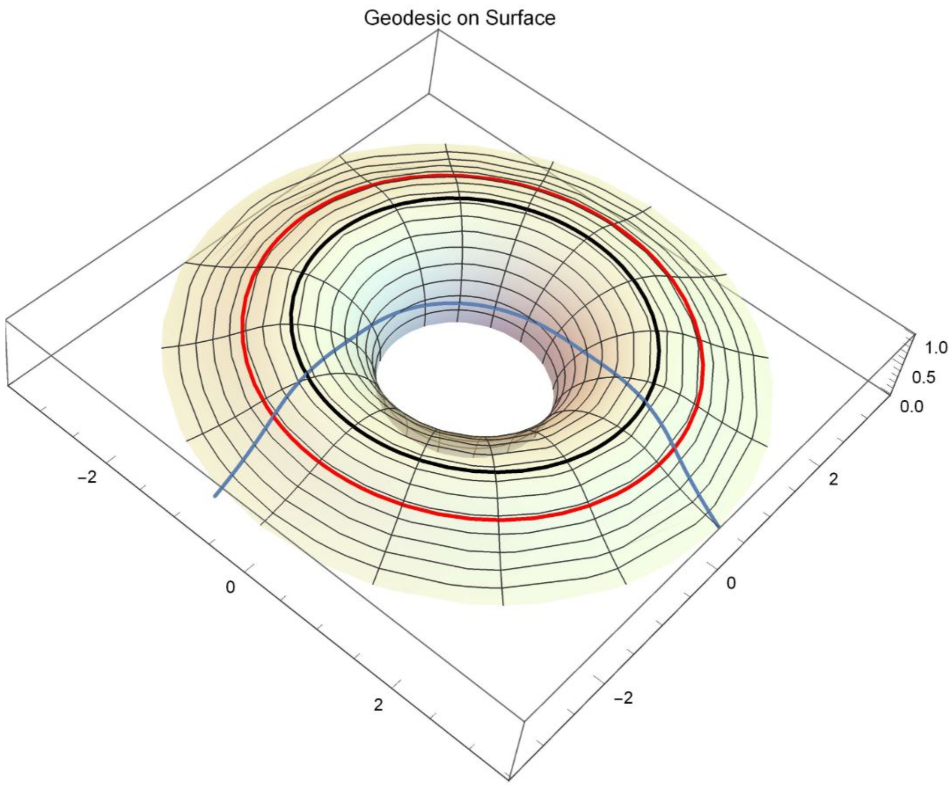

Figure 35: Geodesic with starting angle 0.4π

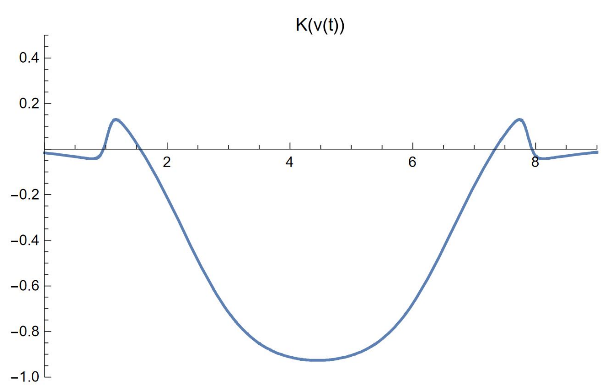

Figure 36: K along geodesic with starting angle 0.4π

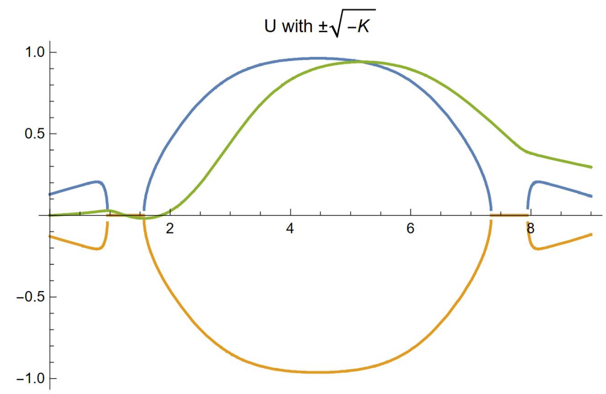

Figure 37: U along geodesic with starting angle 0.4π

7. This geodesic has a starting angle close enough to $\frac{\pi}{2}$ that it follows a trajectory similar to the profile curve (whose starting angle is exactly $\frac{\pi}{2}$). It ends up in the tube, so it only passes through the $K > 0$ region once, and quickly. The solution to the Riccati equation barely falls during this time, and it remains above $-\sqrt{-K}$, so it is strongly unstable.

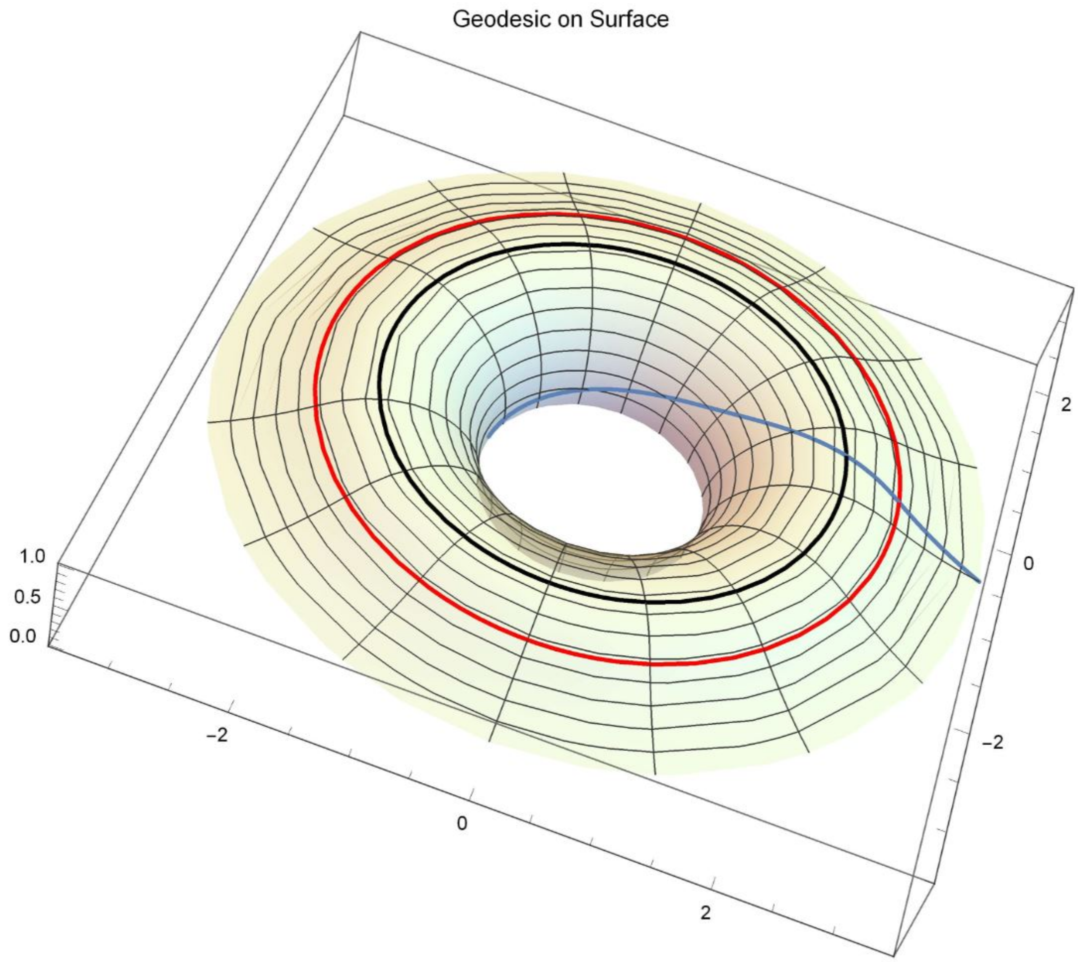

Figure 38: Geodesic with starting angle 0.404π

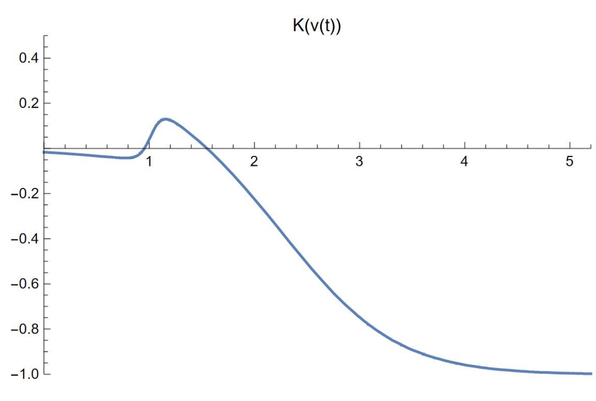

Figure 39: K along geodesic with starting angle 0.404π

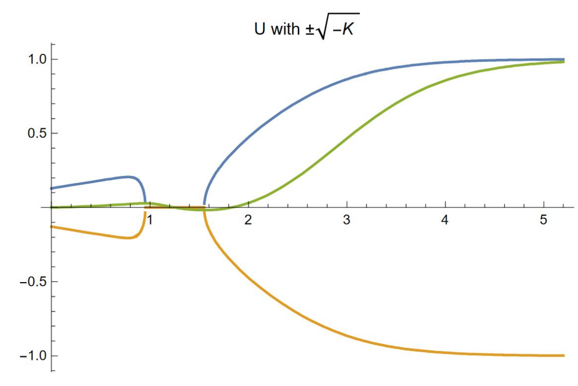

Figure 40: U along geodesic with starting angle 0.404π

Conclusion

We wanted to create a tube that could be used as part of a surface of two connected, concentric hydrant shapes (see Figure 2) that would hopefully have a chaotic system of geodesics. We designed a tube-shaped surface of revolution (see Figure 18) based on the inside of the torus by creating a profile curve. Using the differential equations to solve for geodesics and Clairaut's Relation to choose appropriate starting angles, we considered the behavior of geodesics on the tube. We used the graph of the solution to the Riccati equation to see if the geodesics satisfied the condition of being strongly unstable. Although most of them did, the geodesic with starting angle $\theta_0 = 0.255\pi$ (in Figures 26, 27, and 28) is not strongly unstable. By continuity, there is a family of geodesics with starting angles close to this that all fail the conditions of being strongly unstable. Therefore this proposed tube does not have a chaotic system of geodesics.

In this paper, we explored only one tube based on one profile curve. There are several aspects of the tube that could be changed in an effort to create a chaotic system of geodesics: if the same profile curve structure is used, then the two constant curvatures (we used -1 and 0.3) and the switch point (we used $\frac{2\pi}{3}$) can be changed. Alternatively, a different profile curve structure could be used: one that switches between constant curvatures in a stepwise motion, or one that is not based on constant curvatures to begin with. The underlying question of trying different profile curves, however, is whether a successful one exists at all. Thus, one could try to figure out whether using a small number of tubes to connect two concentric surfaces will always fail to create a chaotic system of geodesics, and why that is.

References

[1] Manfredo doCarmo. *Differential Geometry of Curves and Surfaces*. 1976.

[2] Victor J. Donnay and Charles C. Pugh. *Anosov Geodesic Flows for Embedded Surfaces*. 2000.

[3] Alfred Gray. *Modern Differential Geometry of Curves and Surfaces*. 1993.

[4] Theodore Shifrin. *Differential Geometry: A First Course in Curves and Surfaces*. 2016.

Acknowledgements

I would like to thank my advisor, Victor Donnay, for allowing me to be a part of his long-term research project about chaos and for his patience in figuring out Mathematica together. I am grateful to Josh Sabloff for pushing me to become a better mathematician and for his differential geometry course, which provided me the background knowledge for this thesis. I would also like to thank Veronica Benson-Moore for helping me edit and Eli Fisher for answering all of my LaTeX questions.

Appendix: Chaotic Systems of Geodesics on Surfaces of Revolution

Mathematica Programs - Westley Mildenhall - 28 April 2017

1. Gaussian Curvature of a Surface of Revolution Calculator

Plug in f[v] and g[v] for a surface of revolution into the chart X to calculate the partial derivatives, first (E,F,G) and second (l,m,n) fundamental forms, and matrix dNp.

In the paper, we used this program to find the Gaussian Curvature of the torus.

```
X[u_, v_] := {f[v] Cos[u], f[v] Sin[u], g[v]};
NewNorm[x_] := Sqrt[Sum[x[[i]]^2, {i, 1, Length[x]}]];
duX = D[X[u, v], u];
dvX = D[X[u, v], v];
longN = Cross[duX, dvX];
duuX = D[X[u, v], u, u];
duvX = D[X[u, v], u, v];
dvvX = D[X[u, v], v, v];
NN = longN / NewNorm[longN];
EE = Dot[duX, duX];
FF = Dot[duX, dvX];
GG = Dot[dvX, dvX];
ll = Dot[duuX, NN];
mm = Dot[duvX, NN];
nn = Dot[dvvX, NN];
EEu = D[EE, u];
EEv = D[EE, v];
FFu = D[FF, u];
FFv = D[FF, v];
GGu = D[GG, u];
GGv = D[GG, v];
MatrixForm[A = {{EE, FF}, {FF, GG}}];
MatrixForm[B = {{ll, mm}, {mm, nn}}];
MatrixForm[dNp = Inverse[A].B] // Simplify
```

$$\begin{pmatrix} -\frac{g'[v]}{\sqrt{f[v]^2\left(f'[v]^2+g'[v]^2\right)}} & 0 \\ 0 & \frac{f[v]^3\left(g'[v]\,f''[v]-f'[v]\,g''[v]\right)}{\left(f[v]^2\left(f'[v]^2+g'[v]^2\right)\right)^{3/2}} \end{pmatrix}$$

2. Testing Surfaces of Revolution by changing the space curvature of the profile curve

Replace the SpaceCurvature function with a different one, using as many adjustable parameters as needed (here there are three). This allows you to see what the resulting surface will look like.

In the SpaceCurvature formula here, p1 is the height of the Tanh function, p2 is the switch point (where it goes from the lower to higher constant curvature), and p3 is the lower constant curvature value.

We use a Module to make all of the variables local so that they do not interfere with the other programs (and specifically with the next program, after we decide on a SpaceCurvature to use).

The “First” command used to define the f and v resulting from the NDSolve makes the functions stand-alone rather than being stuck as complicated solutions that can’t be plugged in to graphs and other functions easily.

```
Module[{x, y, θ, SpaceCurvature, ProfileCurve, f, g, p1, p2, p3, u, v, s, A, tt},
 Manipulate[
  SpaceCurvature[x_] := (p1/2) Tanh[(x - p2)/0.1] + (p3 - (p1/2) Tanh[(-p2)/0.1]);
  ProfileCurve =
   NDSolve[{x'[s] == Cos[θ[s]], y'[s] == Sin[θ[s]], θ'[s] == SpaceCurvature[s],
     x[0] == 1, y[0] == 0, θ[0] == π/2}, {x, y, θ}, {s, 0, 2*p2}];
  f[v_] := First[Evaluate[x[tt] /. ProfileCurve]] /. {tt → v};
  g[v_] := First[Evaluate[y[tt] /. ProfileCurve]] /. {tt → v};
  ParametricPlot3D[{f[v] * Cos[u], f[v] * Sin[u], g[v]}, {v, 0, 2*p2}, {u, 0, 2 Pi},
   PlotStyle → Opacity[0.75]], {{p1, 1.3, "Parameter 1"}, 1, 3, Appearance → "Labeled"},
  {{p2, 2 π / 3, "Parameter 2"}, π / 2, π, Appearance → "Labeled"},
  {{p3, -1, "Parameter 3"}, -2, -0.1, Appearance → "Labeled"}]]
```

3. Looking at Geodesics by Starting Angle

3a. Initialization Cells

These initialization cells define:
the space curvature of the profile curve (SpaceCurvature), the resulting f and g (fPC and gPC) for the profile curve, the Gaussian curvature

- SpaceCurvature: the space curvature of the profile curve
- fPC, gPC: the f and g of the profile curve after doing the NDSolve

- KGauss: the Gaussian Curvature K on the resulting surface of revolution
- Surface: the 3D graph of the surface of revolution
- Chart: the function that takes (u,v) and puts it on the surface of revolution (to use when you solve for geodesics (u(t),v(t)))
- vExit, vEnter: lines at constant v-values when K=0

```
SpaceCurvature[x_] :=
  (1.3/2) Tanh[(x - 2 π / 3 + 0.05)/0.1] + (-1 - (1.3/2) Tanh[(-2 π / 3 + 0.05)/0.1]);
ProfileCurve = NDSolve[{x'[s] == Cos[θ[s]], y'[s] == Sin[θ[s]],
    θ'[s] == SpaceCurvature[s], x[0] == 1, y[0] == 0, θ[0] == π / 2}, {x, y, θ}, {s, 0, 4 π/3}];
fPC[v_] := First[Evaluate[x[tt] /. ProfileCurve]] /. {tt → v};
gPC[v_] := First[Evaluate[y[tt] /. ProfileCurve]] /. {tt → v};
D2fPC[v_] := D[fPC[vv], {vv, 2}] /. {vv → v};
KGauss[v_] := -D2fPC[v] / fPC[v];
```

```
Surface = ParametricPlot3D[{fPC[v] * Cos[u], fPC[v] * Sin[u], gPC[v]},
   {v, 0, 3}, {u, -Pi, Pi}, PlotStyle → {LightGreen, Opacity[0.5]}];
Chart[u_, v_] := {fPC[v] * Cos[u], fPC[v] * Sin[u], gPC[v]};
vExit = ParametricPlot3D[Chart[u, 1.5708], {u, 0, 2 Pi}, PlotStyle → Black];
vEnter = ParametricPlot3D[Chart[u, 2.10462], {u, 0, 2 Pi}, PlotStyle → Red];
```

3b. FindRoot Command

To solve for vExit and vEnter, use the “FindRoot” command after looking at a graph to see approximately where the roots are (here it was 1 and 2):

```
FindRoot[KGauss[v], {v, 1}]
```

```
{v → 1.5708}
```

```
FindRoot[KGauss[v], {v, 2}]
```

```
{v → 2.10462}
```

3c. Solving and Plotting Geodesics

After evaluating the initialization cells, we can use the next program to solve and plot geodesics, the Gaussian curvatures along them, and the solution to the Riccati equation along them. Here we have

- angle: the starting angle for the geodesic
  - for starting angles closer to zero, use the first “While” section and leave the second commented out. Use an increment around 0.1.

        - if the starting angle is too close to Pi/2 then the geodesic will never return to v=3 and the first "While" will never be able to finish. If you want to see geodesics with higher angles, then you can comment out using (* x *) the first "While" section and use the second "While" section. For this, use a time=0 and an increment around 0.05.
        - if you want to choose timefinal, comment out both "While" sections and use the "Manually Choose Time." Then the "time" set at the beginning of the program will be used as "timefinal."
    - vinitial: starting value for v, is v[0] in the NDSolve
    - While: this whole command solves for the "timefinal" when v returns to its starting position of 3. Then the rest of the program uses timefinal
        - time: the starting time for the "While." If the program takes too long to finish the "While," you can increase the initial time. If the initial time is higher than the time that v will return to 3, however, it won't work.
    - increment: how precisely timefinal is calculated.
    - RiccatiSoln: the solution to the Riccati equation
    - SqrtK: the function to plot the square root of negative K or zero if K is positive

```
angle = angleC[0];
time = 20;
increment = 0.1;
vinitial = 3;
(*FIRST WHILE*)
(*While[geodesic=NDSolve[{u''[t]+(2fPC'[v[t]])/fPC[v[t]] u'[t]*v'[t]==0,
    v''[t]-fPC[v[t]]*fPC'[v[t]]*(u'[t])^2==0,u[0]==0,v[0]==vinitial,u'[0]==Cos[-angle]/fPC[vinitial],
    v'[0]==Sin[-angle]},{u,v},{t,0,time}];
 ugeo[t_]:=First[Evaluate[u[aa]/.geodesic]]/.{aa→t} ;
 vgeo[t_]:=First[Evaluate[v[aa]/.geodesic]]/.{aa→t};
 vgeo[time]≤vinitial,time=time+increment];timefinal=time;*)
(*SECOND WHILE*)
(*While[geodesic=NDSolve[{u''[t]+(2fPC'[v[t]])/fPC[v[t]] u'[t]*v'[t]==0,
    v''[t]-fPC[v[t]]*fPC'[v[t]]*(u'[t])^2==0,u[0]==0,v[0]==vinitial,u'[0]==Cos[-angle]/fPC[vinitial],
    v'[0]==Sin[-angle]},{u,v},{t,0,time}];
 ugeo[t_]:=First[Evaluate[u[aa]/.geodesic]]/.{aa→t} ;
 vgeo[t_]:=First[Evaluate[v[aa]/.geodesic]]/.{aa→t};
 vgeo[time]≥0.05,time=time+increment];timefinal=time;*)
(*MANUALLY CHOOSE ITME*)
geodesic = NDSolve[{u''[t] + (2 fPC'[v[t]])/fPC[v[t]] u'[t] * v'[t] == 0,
    v''[t] - fPC[v[t]] * fPC'[v[t]] * (u'[t])^2 == 0,
    u[0] == 0, v[0] == vinitial, u'[0] == Cos[-angle]/fPC[vinitial],
    v'[0] == Sin[-angle]}, {u, v}, {t, 0, time}];
```

```
ugeo[t_] := First[Evaluate[u[aa] /. geodesic]] /. {aa → t} ;
vgeo[t_] := First[Evaluate[v[aa] /. geodesic]] /. {aa → t};
timefinal = time;
(*Rest of Program*)
Print@
  ParametricPlot[{ugeo[t], vgeo[t]}, {t, 0, timefinal}, PlotLabel → "Geodesic in u-v"];
Print@Plot[KGauss[vgeo[t]], {t, 0, timefinal},
   PlotRange → {{0, timefinal}, {-.5, .5}}, PlotLabel → "K(v(t))"];
Show[vEnter, vExit, ParametricPlot3D[Chart[ugeo[t], vgeo[t]], {t, 0, timefinal}],
 Surface, PlotRange → All, PlotLabel → "Geodesic on Surface"]
RiccatiSoln = NDSolve[{U'[t] == -KGauss[vgeo[t]] - (U[t])^2, U[0] == 0},
   U, {t, 0, timefinal}];
URic[t_] := First[Evaluate[U[tt] /. RiccatiSoln]] /. {tt → t};
SqrtK[t_] :=
  Piecewise[{{√(-KGauss[vgeo[t]]), KGauss[vgeo[t]] < 0}, {0, KGauss[vgeo[t]] > 0}}];
Print@Plot[{SqrtK[t], -SqrtK[t], URic[t]}, {t, 0, timefinal}, PlotLabel → "U with ±√-K"]
```

4. Clairaut Angle Solver

Using Clairaut’s Relation, we make this function that tells us the starting angle to use for a geodesic if we want it to turn around at v=vturn. We find the angles for v=1.5708 and v=2.10463, where K is zero. Then we find the angle for v=0 which should give us a geodesic asymptotic to the center of the tube.

```
angleC[vturn_] := ArcCos[fPC[vturn] / fPC[vinitial]]
```

```
angleC[1.5708]
```

0.93305

```
angleC[2.10462]
```

0.726157

```
angleC[0]
```

1.26852

5. Jacobi and Riccati Constant Curvature Cases

We look at the solutions to the Jacobi equation in three cases of initial conditions. Then we plot these J graphs with their corresponding U graphs using the definition $U = \frac{J'}{J}$. The “pos/neg/zero” in the function names refer to the Gaussian curvature.

Initial Conditions (a)

```
apos = DSolve[{J''[t] + J[t] == 0, J[0] == 0, J'[0] == 1}, J, t];
aPos[t_] := First[Evaluate[J[tt] /. apos]] /. tt → t;
aneg = DSolve[{J''[t] - J[t] == 0, J[0] == 0, J'[0] == 1}, J, t];
aNeg[t_] := First[Evaluate[J[tt] /. aneg]] /. tt → t;
azero = DSolve[{J''[t] == 0, J[0] == 0, J'[0] == 1}, J, t];
aZero[t_] := First[Evaluate[J[tt] /. azero]] /. tt → t;
```

Initial Conditions (b)

```
bpos = DSolve[{J''[t] + J[t] == 0, J[0] == 1, J'[0] == 0}, J, t];
bPos[t_] := First[Evaluate[J[tt] /. bpos]] /. tt → t;
bneg = DSolve[{J''[t] - J[t] == 0, J[0] == 1, J'[0] == 0}, J, t];
bNeg[t_] := First[Evaluate[J[tt] /. bneg]] /. tt → t;
bzero = DSolve[{J''[t] == 0, J[0] == 1, J'[0] == 0}, J, t];
bZero[t_] := First[Evaluate[J[tt] /. bzero]] /. tt → t;
```

Initial Conditions (c)

```
cpos = DSolve[{J''[t] + J[t] == 0, J[0] == 1, J'[0] == -1}, J, t];
cPos[t_] := First[Evaluate[J[tt] /. cpos]] /. tt → t;
cneg1 = DSolve[{J''[t] - J[t] == 0, J[0] == 1, J'[0] == -0.9}, J, t];
cNeg1[t_] := First[Evaluate[J[tt] /. cneg1]] /. tt → t;
cneg2 = DSolve[{J''[t] - J[t] == 0, J[0] == 1, J'[0] == -1.1}, J, t];
cNeg2[t_] := First[Evaluate[J[tt] /. cneg2]] /. tt → t;
czero = DSolve[{J''[t] == 0, J[0] == 1, J'[0] == -1}, J, t];
cZero[t_] := First[Evaluate[J[tt] /. czero]] /. tt → t;
```

Graphs of J and U together

```
Plot[{cNeg2'[t]/cNeg2[t], cNeg2[t], -1}, {t, 0, 3}, PlotRange → {{0, 3}, {-6, 1}},
 PlotLegends → {U, J, "-√-K "}, PlotLabel → "K=-1 with J[0]=1 and J'[0]=-1.1"]

Plot[{bPos'[t]/bPos[t], bPos[t]}, {t, 0, 3}, PlotLegends → {U, J}, PlotLabel → "K=1 Case"]
```

```
Plot[{bNeg'[t]/bNeg[t], bNeg[t]}, {t, 0, 3}, PlotLegends → {U, J}, PlotLabel → "K=-1 Case"]
```

```
Plot[{bZero'[t]/bZero[t], bZero[t]}, {t, 0, 3}, PlotLegends → {U, J},
 PlotLabel → "K=0 Case", PlotStyle → {Thickness[.009], Thickness[.005]}]
```

6. Plotting Multiple Geodesics at once

Here is an example of how to plot a set of geodesics on the torus with a range of starting angles by using the “Table” command. This program uses:

- Torus: the chart of the torus
- Tplot: the 3D plot of the torus
- ftorus: the f value for the profile curve (f[v],g[v])
- timeT: how long to solve the geodesic for
- angleT: the angle coefficient for the “Table” command. This is multiplied by n, which is given a range of integer values at the very end of the program.
- vinitial: the initial v-value for the geodesic. Since u is the rotation, it doesn’t matter, but v changes the height on the torus.

```
Torus[u_, v_] := {(2 + Cos[v]) Cos[u], (2 + Cos[v]) Sin[u], Sin[v]};
Tplot = ParametricPlot3D[Torus[u, v],
   {u, 0, 2 Pi}, {v, 0, 2 Pi}, PlotStyle → Opacity[0.5], Mesh → None];
ftorus[v_] := 2 + Cos[v];
timeT = 30;
angleT = Pi / 12;
vinitialT = 0;
Table[geoT = NDSolve[{u''[t] + (2 ftorus'[v[t]])/ftorus[v[t]] u'[t] * v'[t] == 0,
    v''[t] - ftorus[v[t]] * ftorus'[v[t]] * (u'[t])^2 == 0, u[0] == 0, v[0] == vinitialT,
    u'[0] == Cos[angleT * n]/ftorus[vinitialT], v'[0] == Sin[angleT * n]}, {u, v}, {t, 0, timeT}];
 ugeoT[t_] := First[Evaluate[u[aa] /. geoT]] /. {aa → t};
 vgeoT[t_] := First[Evaluate[v[aa] /. geoT]] /. {aa → t};
 geoTplot = ParametricPlot3D[
   {Torus[ugeoT[t], vgeoT[t]]}, {t, 0, timeT}, PlotStyle → Thickness[0.008]];
 Show[Tplot, geoTplot], {n, 0, 6}]
```