EVERYWHERE CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTIONS

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ABSTRACT. Here I discuss the use of everywhere continuous nowhere differentiable functions, as well as the proof of an example of such a function. First, I will explain why the existence of such functions is not intuitive, thus providing significance to the construction and explanation of these functions. Then, I will provide a specific detailed example along with the proof for why it meets the requirements not only of being continuous over the whole real line but also of not having a derivative at any real number. I will next analyze the proof and compare the function to another example of a continuous everywhere nowhere differentiable function in order to pull out how these functions sidestep intuition.

1. Continuity and Differentiability: The Failure of Intuition

In mathematics, not all concepts which seem intuitive at first are indeed correct. Fortunately, often having enough experience working in a field will expose a person to the field's intricacies, and thus change the person's intuition or at least make it weary of certain ideas. On the other hand, sometimes the counter-examples to an idea can be rare or contrived, preventing the intuition from naturally adapting and leaving the person vulnerable to making a fatal assumption. Hence, with such concepts, going out of one's way to remember a counter-example can keep one on one's toes, and understanding it will give one a more nuanced understanding of the ideas involved.

The relationship between the continuity and differentiability of a function contains an example of a deceptively intuitive link, though a number of intuitive ideas about the relationship apply. In one direction, the straight forward idea holds: if a function is differentiable at a point, then it must be continuous there as well. It is also not counter-intuitive why the opposite implication, that a derivatives exist wherever a function is continuous, does not hold. Often one of the first things a calculus student learns is that if a function exhibits a 'sharp turn' that the derivative does not exist, and the absolute value function provides an easily conceptualized example of this concept. At 0 the left handed derivative of the absolute value function equals -1, as the function in question is a simple linear function with a slope of -1, and similarly the right handed derivative equals 1. Because derivatives are based on limits, if the right handed derivative and the left handed derivative are not equal then the derivative as a whole cannot exist. Hence, though the absolute value function is continuous at 0, it is not differentiable there. Even without the formalism, one can see that at 'sharp turns' the tangent line wants to have two distinct slopes: the slope as conceived from the left and the slope as conceived from the right. The logical choice is to not choose between them and to declare the derivative non-existent, as is done formally.

The deception occurs in that for continuous functions failure of differentiation seems rare. Given a continuous function, one typically assumes that the derivative exists at most points, though the derivative could exist nowhere. A couple of factors fuel the incorrect intuition. For one, a number of the functions that math students first learn to work with behave nicely with respect to the derivative over their domain. These include polynomial, rational, trigonometric, exponential, and logarithmic functions, which in fact are differentiable everywhere on their domain. Of the power functions, which have the form $f(x) = x^a$ for some $a \in \mathbb{R}$, only have points for which they are not differentiable for $0 < a < 1$. This is because by the power rule of differentiation we will have $f'(x) = ax^{a-1}$, if $a \neq 0$. If $a \geq 1$, the derivative continues to exist everywhere. If $a < 0$, it does not exist at 0, but that was not a part of the functions domain. For $0 < a < 1$, then, f exists at everywhere and is differentiable everywhere but at 0. In the case that $a = 0$, we merely have $f(x) = 1$, which has derivative 0 everywhere.

Even when these commonly used functions have continuous points without a derivative, they don't typically have many. For example, the absolute value function and the power functions described above only lack a derivative at a continuous point at 0. One can think of methods such as forcing periodicity to increase the number of continuous non-differentiable points given that one such point exists, but even that will not force the number of non-differentiable points into the uncountable range if it were not there already. Hence these functions give the idea that non-differentiability is actually relatively rare for continuous functions.

Moreover, both the ideas of a function's continuity and differentiability relate, at least intuitively to ideas of smoothness. Before learning the formal definition of continuity, often students hear the idea that if one can draw a function without picking up one's pencil then it is continuous. As stated, this intuitive test manifests a logically correct idea. If a function is not continuous at a point c then no mater how small of an interval one creates around c, there will always be points at least ϵ away for some $\epsilon > 0$. Hence, when drawing the function one would need to move ϵ away from $f(c)$, for all intents and purposes, hence requiring the pencil to be picked up. Unfortunately, until students reach more formal classes such as Real Analysis, students typically learn how to find discontinuities in a few types of functions and that elsewhere these functions are continuous. These functions are the one's noted above: polynomial, rational, trigonometric, exponential, and logarithmic functions. Hence, students do not confront early on functions that are nearly impossible to draw, let alone draw without picking up one's pencil in terms of continuity. Yet, these functions can still have points at which they are continuous. the Thomae function, which provides an example of a function continuous on the irrationals, where it has non-zero values at all the rational numbers and is valued 0 at the irrationals 1. This would be quite impractical to draw because both the rationals and the irrationals are dense in $\mathbb{R}$ [1]. Without exposure to such functions, continuity can become conflated with "easy to draw", and even the example above is not continuous at countably many points.

The derivative has even deeper connotations of smoothness. In order for a function to have a derivative at a point c, not only must the function be continuous at c and hence providing all of the connotations related to that, but one must be able to approximate the function linearly at that point. Because linear functions are particularly smooth, the existence of such an approximation hints that the original function must be smooth as well. In fact, the formula for the derivative provides a formal method for finding such an approximation by taking the limit of the slope between c and some other point as the point approaches c. When drawing a function without picking the pencil off the paper, one only has the option to choose which direction the pencil will go. At points where the function is differentiable one can move the pencil for a short period of time in the direction of the tangent line, and, if necessary, changing the direction as one goes to create the illusion of curvature. At a point where the derivative does not exist, one can quickly change the direction of the stroke, but the hand will still go for a short period of time in some direction. On that interval, the function will have a derivative, illustrating the difficulty of drawing and perhaps then conceptualizing a continuous function without a derivative. In fact, when the software Mathematica attempts to graphically display known examples of everywhere continuous nowhere differentiable equations such as the Weierstrass function or the example provided in Abbot's textbook, Understanding Analysis, the functions appear to have derivatives at certain points. See figures 1 and 2 for examples. Hence, a function's continuity can hide its non-differentiability.

2. Abbot's Example: The Function and Proof

Abbot provides an example of an everywhere continuous nowhere differentiable equation, though it does not give all the details for the proof of nowhere differentiability, which itself is a bit complicated. However, because the idea of an everywhere continuous nowhere differentiable equation is so counter intuitive, we will provide a thorough proof here in order to highlight exactly where the intuition breaks down.

To construct the function, we will first consider a simpler function h, where for $-1 \leq x \leq 1$, $h(x) = |x|$ which is then made periodic by the condition that $h(x + 2) = h(x)$. For all $y \in R$ there exists a $2k \in 2\mathbb{Z}$, such that $y + 2k \in [-1, 1]$ because the even integers are spaced 2 apart and have no lower or upper bounds in $\mathbb{R}$.

Moreover, we have $h(x+2n) = h(x)$ for all $n \in \mathbb{N}$. For $n = 1$ this is just h$(x+2) = h(x)$ as above. Assuming inductively that $h(x+2n) = h(x)$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned} h(x+2(n+1)) &= h(x+2n+2) \\ &= h(x+2n) \\ &= h(x) \end{aligned}$$

as needed.

Hence, for all even integers m, $h(m) = 0$ and for all odd integers n, $h(n) = 1$. In fact 1 bounds this function from above, while 0 bounds the function from below as between any two odd integers, the function will mimic the values between -1 and 1. These themselves are bounded by 0 and 1. Any point between two integers n and $n+1$, the function is modeled by a line connecting $(n, h(n))$ and $(n+1, h(n+1))$ The function will then be continuous over all of $\mathbb{R}$. Therefore, for any $x \in \mathbb{R} \backslash \mathbb{Z}$, the derivative of h at x exists and is 1 if the previous integer was even and -1 if the previous integer was odd.

At any integer $x \in \mathbb{Z}$, the right handed derivative and left handed derivative at x will correspond to the slopes of the lines on the right and left hand sides of x. The line to the right of x will have slope ± 1 depending on whether x is odd or even. However, in either case the integer immediately to the left of x will be even if x is odd and odd if x is even. So the slope to the left of x will have the opposite value as the slope to the right of x, but in either case the derivative will not exist.

We will then show that g where

$$g(x) = \sum_{i=0}^{\infty} \frac{h(2^i x)}{2^i}$$

is continuous everywhere but differentiable nowhere. Because we will be discussing g at various stages of its construction, we will introduce additional notation. In particular,

$$\begin{aligned} f_i(x) &= \frac{h(2^i x)}{2^i} \\ g_n(x) &= \sum_{i=0}^{n} f_i(x) \end{aligned}$$

Moreover, because we start the series of functions with f_0, considering 0 as a part of the natural numbers will simplify notation without causing any difficulties. We will now note some useful properties of f_i and g_n. For example, $f_i(x)$ scales the h by $\frac{1}{2^i}$ both horizontally and vertically, so $f_i(x)$ equals 0 if $2^i x$ is an even integer and equals $\frac{1}{2^i}$ if $2^i x$ is an odd integer. These points are connected by lines, so f_i is continuous every where. By the chain rule the derivative of f is $f_i'(x) = \frac{h'(2^i x)2^i}{2^i} = h'(2^i x)$, which is just h' scaled horizontally by $\frac{1}{2^i}$. Thus, if $2^i x \in Z, h'(x)$ does not exist, and for $x \in \left(\frac{z}{2^i}, \frac{z+1}{2^i}\right)$ where $z \in \mathbb{Z}$, if z is even $f_i'(x) = 1$ and if z is odd $f_i'(x) = -1$.

If $f_i(x) = 0$ for some $i \in \mathbb{N}$, then $2^i x \in 2\mathbb{Z}$. $f_{i+1}(x) = \frac{f_i(2x)}{2}$, and $2\left(2^i x\right)$ will also be in $2\mathbb{Z}$, so $fi + 1(x) = 0$. Therefore, given an n such that $fn + 1(x) = 0$, we have that $g(x) = g_n(x)$. Using this we can say that because $f_1(0) = h(2(0)) = h(2) = 0$, $g(0) = g_0(0) = h(0) = 0$. Moreover, for $x \in [-\frac{1}{2^n}, \frac{1}{2^n}]$, $f_n(x) = |x|$. Also, g will have a period of 2 just as h did, because for all $i \in \mathbb{N}$, $f_i(x+2) = f_i(x)$. f_0 is just h for which this is assumed. Assuming inductively that $f_i(x+2) = f(x)$ for some $i \in \mathbb{N}$, we know that $f_{i+1}(x) = \frac{f_i(2x)}{2}$. Therefore,

$$\begin{aligned} f_{i+1}(x+2) &= \frac{f_i(2x+4)}{2} \\ &= \frac{f_i(2x+2)}{2} \\ &= \frac{fi(2x)}{2} \\ &= f_{i+1}(x) \end{aligned}$$

Hence, $f_i(x+2) = f(x)$ holds for all $i \in \mathbb{N}$ by induction. Therefore,

$$\begin{aligned} g(x+2) &= \sum_{i=0}^{\infty} f_i(x+2) \\ &= \sum_{i=0}^{\infty} f_i(x) \\ &= g(x) \end{aligned}$$

In order to show that g is continuous, one only needs to apply the $M-$test. $h(2^i x)$ is bounded between 0 and 1 for all $i \in \mathbb{N}$, so f_i will be bounded between 0 and $\frac{1}{2^i}$ for all $i \in \mathbb{N}$. $\sum_{i=0}^{\infty} \frac{1}{2^i}$ is a geometric series with initial value 1 and a ratio of $\frac{1}{2}$. Therefore it converges, so the series $\sum_{i=0}^{\infty} f_i(x)$ converges uniformly. Because all f_i are continuous over all of $\mathbb{R}$ we then have that g is continuous over all of $\mathbb{R}$ [1].

The more difficult claim is that g is differentiable nowhere. To do this one does not prove that g is not differentiable at all $x \in \mathbb{R}$ all at once, but instead breaks the x into cases which build on each other.

First, we start simple and prove that g is not differentiable at 0. Consider the sequence $x_m = \frac{1}{2^m}$. Because $\lim x_m = 0$, if g is to be differentiable at 0,

$$\begin{aligned} \lim \frac{g(x_m) - g(0)}{x_m - 0} &= \lim \frac{g(x_m)}{x_m} \\ &= \lim \frac{g(2^{-m})}{2^{-m}} \\ &= \lim 2^m g(2^{-m}) \\ &= L \end{aligned}$$

for some $L \in \mathbb{R}$. However, $2^{m+1}\left(\frac{1}{2^m}\right) \in 2\mathbb{Z}$, so $f_{m+1}(x_m) = 0$ and thus $g(x_m) = g_m(x_m)$. Therefore,

$$\begin{aligned}
2^m g(2^{-m}) &= 2^m g_m(2^{-m}) \\
&= 2^m \sum_{i=0}^{m} f_i(2^{-m}) \\
&= 2^m \sum_{i=0}^{m} 2^{-m} \\
&= \sum_{i=0}^{m} 1 \\
&= m + 1
\end{aligned}$$

, so $\lim \frac{g(x_m)-g(0)}{x_m-0} = \lim m+1$, which diverges. Hence, g is not differentiable at 0. Because g has a period of 2, this then means that g is not differentiable at any point in $2\mathbb{Z}$.

Next, we will show that g is not differentiable at 1. First note that for $x \in [\frac{1}{2}, 1)$, $g_1'(x) = h'(x) + h'(2x) = 1 - 1 = 0$ and that for $x \in (1, \frac{3}{2}]$, $g_1'(x) = h'(x) + h'(2x) = -1 + 1 = 0$. Therefore, because

$$\begin{aligned}
g_1(.5) &= h(.5) + .5h(1) \\
&= .5 + .5 \\
&= 1 \\
g_1(1) &= h(1) + .5h(2) \\
&= 1 + 0 \\
&= 1 \\
g1(1.5) &= h(1.5) + .5h(3) \\
&= .5 + .5 \\
&= 1
\end{aligned}$$

for all $x[\frac{1}{2}, \frac{3}{2}]$, $g_1(x) = 1$. Furthermore, since $f_1(1) = 0$, we know that $g(1) = f_0(1) = 1$

Now consider the sequence $x_m = 1 - \frac{1}{2^{m+1}}$. First, $2^{m+2}x_m = 2^{m+2} - 2 \in 2\mathbb{Z}$, so $f_{m+2}(x_m) = 0$, and thus $g(x_m) = g_{m+1}(x_m)$. Now, $\lim x_m = 1$, so if the derivative of g exists at 1 it will equal,

$$
\begin{aligned}
\lim \frac{g(x_m) - g(1)}{x_m - 1} &= \lim \frac{g(1 - \frac{1}{2^{m+1}}) - 1}{-2^{-m-1}} \\
&= \lim -2^{m+1} \sum_{i=0}^{m+1} f_i(1 - 2^{-m-1}) - 1 \\
&= \lim -2^{m+1} \sum_{i=2}^{m+1} f_i(1 - 2^{-m-1}) \\
&= \lim -2^{m+1} \sum_{i=2}^{m+1} \frac{h(2^i - 2^{i-m-1})}{2^i} \\
&= \lim -2^{m+1} \sum_{i=2}^{m+1} \frac{h(-2^{i-m-1})}{2^i} \\
&= \lim -2^{m+1} \sum_{i=2}^{m+1} 2^{-m-1} \\
&= \lim \sum_{i=2}^{m+1} -1 \\
&= \lim -m
\end{aligned}
$$

However, this diverges, so g is not differentiable at 1. Combining this with our earlier conclusion that g is not differentiable at 0 gives that g is not differentiable at any integer.

We will now look at points that have the form $x = \frac{z}{2^N}$ for $N \in \mathbb{N}$ and $z \in \mathbb{Z}$. For simplicity, we assume that this fraction is reduced. If x is an integer, we already know that g is not differentiable there. Otherwise, we have that z is odd and $N \geq 1$. Hence, for all $n \leq N01$, $2^n \frac{z}{2^N} \notin 2Z$, so f_n is differentiable at x and thus g_{N-1} is differentiable at x. g will then be differentiable at x if $g - g_{N-1}$ is differentiable at x. Note then that,

$$
\begin{aligned}
(g - g_{N-1})(x) &= \sum_{i=N}^{\infty} f_i(x) \\
&= \sum_{i=N}^{\infty} \frac{h(2^i x)}{2^i} \\
&= \sum_{i=0}^{\infty} \frac{h(2^{i+N} x)}{2^{i+N}} \\
&= \frac{g(2^N x)}{2^N}
\end{aligned}
$$

Dividing by 2^N does not affect differentiability, so in order for g to be differentiable at x, g must be differentiable at $2^N x = 2^N \frac{z}{2^N} = z \in \mathbb{Z}$, which we already established was not differentiable.

Finally, we we look at points $x \in \mathbb{R}$ that do not have this form. For each $N \in \mathbb{N}$, there will exist a $z_N \in \mathbb{Z}$ such that $\frac{z_N}{2^N} < x < \frac{z_N+1}{2^N}$. As such we can create two sequences (x_m) and (y_m) out of these where $xm = \frac{z_m}{2^m}$ and $ym = \frac{z_m+1}{2^m}$, where $\lim x_m = x = \lim y_m$. In fact, these describe the end points of the line on which x sits in f_m, and that line will be contained in a line on f_n for all $n \leq m$. Therefore, the portion between x_m and y_m on g_m will be linear. Because x does not have the form $\frac{z}{2^N}$ for some $N \in \mathbb{N}$, for no $n \in \mathbb{N}$ will $2^n x$ be an integer, so f_n will be differentiable at x for all $n \in \mathbb{N}$. Hence, g_n will be differentiable at x for some all $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}|g'_{m+1}(x) - g'_m(x)| &= \left|\sum_{i=0}^{m+1} f'_i(x) - \sum_{i=0}^{m} f'_i(x)\right| \\ &= |f'_{m+1}(x)| \\ &= 1\end{aligned}$$

Hence, the sequence $g'_m(x)$ is not Cauchy and will not converge.

We will now compare $g'_m(x)$, which will be the slope of the line on which x sits in g_m to the values $\frac{g(x_m)-g(x)}{x_m-x}$ and $\frac{g(y_m)-g(x)}{y_m-x}$. First we know that $2^{m+1}x_m = 2z \in 2\mathbb{Z}$ and $2^{m+1}y_m = 2z + 2 \in 2\mathbb{Z}$, so $f_{m+1}(x_m) = f_{m+1}(y_m) = 0$. Hence, $g(x_m) = g_m(x_m)$ and $g(y_m) = g_m(y_m)$. Furthermore, because x is differentiable at every f_m it is non-zero all those values, so $g(x) = g_m(x) + a_m$ for some $a_m > 0$,since f_m will always be non-negative. Because $g_m(x)$ is linear between x_m and y_m, the slope of the line between x and these two points will be $g'_m(x)$. From this we have that

$$\begin{aligned}\frac{g(x_m) - g(x)}{x_m - x} &= \frac{g_m(x_m) - (g_m(x) + a_m)}{x_m - x} \\ &= \frac{g_m(x_m) - g_m(x)}{x_m - x} - \frac{a_m}{x_m - x} \\ &= g'_m(x) - \frac{a_m}{x_m - x} \\ &> g'_m(x)\end{aligned}$$

and

$$\begin{aligned}\frac{g(y_m) - g(x)}{y_m - x} &= \frac{g_m(y_m) - (g_m(x) + a_m)}{y_m - x} \\ &= \frac{g_m(y_m) - g_m(x)}{y_m - x} - \frac{a_m}{y_m - x} \\ &= g'_m(x) - \frac{a_m}{y_m - x} \\ &< g'_m(x)\end{aligned}$$

which gives us that

$$\frac{g(y_m) - g(x)}{y_m - x} < g'_m(x) < \frac{g(x_m) - g(x)}{x_m - x}$$

If $g'(x)$ is to exist then $\lim \frac{g(y_m)-g(x)}{y_m-x} = g'(x) = \lim \frac{g(x_m)-g(x)}{x_m-x}$. The squeeze theorem would then tell us that $g'_m(x)$ converges to $g'(x)$, but we have already established that this is impossible, providing a contradiction. As this was the last case of points for which we had to determine the differentiability, we have now proven that g is differentiable nowhere.

3. Analysis and Comparison

Now that we have proven that our function g is continuous everywhere but differentiable nowhere, we should analyze the proof to better understand how it works and how it sidestepped our intuition. We can also see how those ideas can appear in other everywhere continuous nowhere differentiable equations such as the Weierstrass functions, the first well known example of such a function, though not the first example discovered. The Weierstrass functions are a family of functions which have the form

$$g(x) = \sum_{i=0}^{\infty} a^i cos(b^i x)$$

where a and b satisfy certain properties. These were originally $a < 1$ and $ab > 1 + \frac{3\pi}{2}$ and b being an odd integer greater than 1, though later the restrictions were loosened to $a < 1$, $b > 1$, and $ab \leq 1$ [2]. One such function is shown in figure 2.

One should note that we avoided making the assumption that because the derivatives do not exist at the integers for any f_i, they do not exist at their sum. g_1 on $[\frac{1}{2}, \frac{3}{2}]$ provides a counter example because over this interval g_1 equals a constant, 1, so the derivative of g_1 exists at 1 and is 0. However, both f_0 and f_1 do not have derivatives at 1. One can also easily imagine an infinite sequence of functions h_n valued as 0 for $x < 0$ and as $e^{-i}x$ for $x > 0$. No h_n has a derivative at 0 because the left handed derivative is 0 and the right handed derivative is $e^{-n} \neq 0$. The limit function here is simply the 0 function, which has a derivative at 0, namely 0. Hence the derivative of a limit function is not necessarily the limit of the derivatives. The Weierstrass functions therefore provide a good example where the individual functions being added together are differentiable everywhere but the end result is differentiable nowhere.

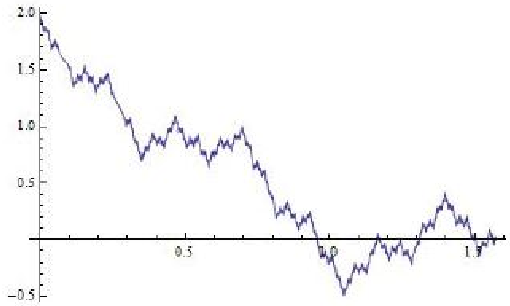
Another thing we should note here is that the function appears smooth at points because the action that causes the breaks in the derivatives occur on the small scales, which our brain can find hard to picture. Figure 3 helps illustrate this. Adding f_i to a function changes the derivative of a function just as much for large i as for small i in that in any case the derivative's change will by 1 or -1 if the derivative of f_i exists at that point. However, figures 3 and 4 illustrate that for large i, the human mind will not notice the change nearly as much because they occur over such miniscule scales. In fact figure 4 shows that adding f_7 to a function k, such as $\sin(3x)$, is barely distinguishable from k, while adding f_3 produces clear changes. Hence, while one typically thinks that a functions derivative is clearly understandable from picture and that having an smooth graph makes a function likely to be differentiable, this is not necessarily the case.

This proof also demonstrates a way that a function can fail to have a derivative at a point without showing a sharp turn. For example, the slopes of the secant lines could tend towards infinity as we showed at the points 0 and 1. Now, our proof does not show that this is what happens at 0 and 1; we only looked at one sequence of points heading towards these values and others could act difierently. The function f where $f(x) = x\frac{1}{3}$ provides an example of this occurring at 0. f, here, is the inverse of the function that cubes x, which by the power rule has a slope of 0 at 0. Therefore, the tangent line of f would have to be vertical, showing that the secant lines would head towards infinity as the second point chosen approached 0. Another thing to note here is that the example highlights the fractal like properties of continuous everywhere and differentiable nowhere equations. The comparison comes naturally because while they are continuous they can never become simple enough to be approximated by a tangent line. However, this example brings the self-similarity principle to the foreground because at each step we add a scaled down copy of our original equation. Interestingly a similar thing occurs in the Weierstrass functions, though the scaling may not be exactly the same vertically and horizontally due to fewer restrictions on a and b.

4. Conclusions

While the existence of derivatives can provide a lot of useful information, one should not assume that a function will be differentiable at most or even any points even if the function is continuous. Fortunately, while the function given by Abbott and the Weierstrass function seem slightly construed for the task, they are formed using simple basic building blocks, such as the absolute value function and the cosine function, and thus are not that difficult to conceive. These functions make use of fractal like designs to keep complexity at all scales and thus prevent linear approximation at any scale and thus the existence of derivatives. At the same time, proving that such functions are actually not differentiable anywhere can require a careful mind to split apart what exactly prevents the derivative from existing at each point.

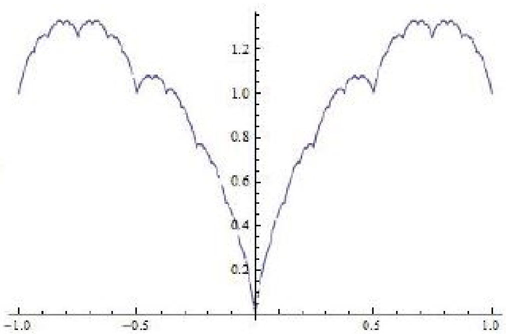
Figure 1: A Weierstrass function using a = .5 and b= 3.
While this is actually only a partial sum of the series that creates the function, including more terms does not actually improve the graph plotted here.




```
= Plot[{Sn[x, 9]}, {x, -1, 1}]
```

Figure 2: the function given in Abbot's texbook.
This is actually only a partial sum to the series that creates the function, but including more terms does not improve the image plotted here.

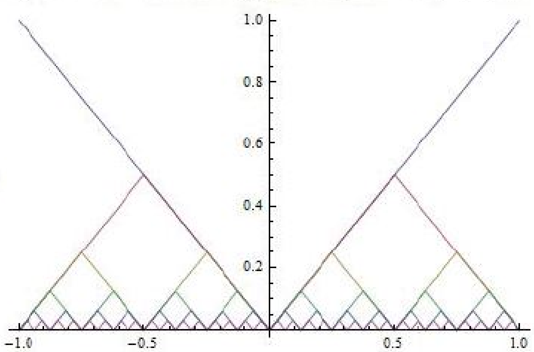
|=




```
In[13]:= Plot[{Hn[x, 0], Hn[x, 1], Hn[x, 2], Hn[x, 3], Hn[x, 4], Hn[x, 5]}, {x, -1, 1}]
```

Figure 3: The Functions f_i(x) for i equal to 0, 1, 2, 3, 4, and 5

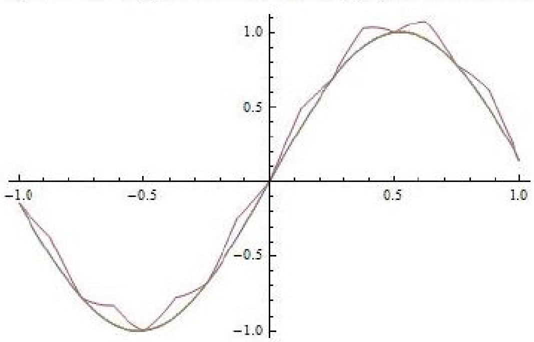
Out[13]=




```
In[21]:= Plot[{Sin[3 x], Hn[x, 3] + Sin[3 x], Hn[x, 7] + Sin[3 x]}, {x, -1, 1}]
```

Figure 4: Letting k be the function sin(3x), this shows the functions k, k + f_3, and k + f_7

Out[21]=



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