An Introduction to the Riemann Hypothesis

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Abstract

Here we discuss the most famous unsolved problem in mathematics, the Riemann hypothesis. We construct the zeta function and its analytic continuation. We note symmetries of zeta and prove that all nontrivial zeros lie in the critical strip, the strip in the complex plane with real part between zero and one inclusive. Finally, we make the connection from the nontrivial zeros to the primes and state some properties of these zeros.

1 Introduction

"...there is a sense in which we can give a one-line non technical statement of the Riemann Hypothesis: 'The primes have music in them'" [3]. Although never passing as a formal definition, this statement by M. Berry and J.P. Keating does express the mathematical community's common association of the primes with magic or perhaps beauty. And at the heart of the primes' magic lies the Riemann Hypothesis. In some sense, the proof of the Riemann hypothesis will be the proof of the existence of magic; that is, of course, if a proof is possible.

The human fascination with primes began in antiquity with Euclid's proof of the infinity of primes. Major headway was made in the understanding of the primes when, in 1793, Gauss conjectured the Prime Number Theorem [7], which asserts that the number of primes no greater than a given number x is asymptotically equivalent to $\frac{x}{\ln x}$ or to $\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$. That is to say, if $\pi(x)$ is the number of primes less than or equal to x, [2, 3, 4]

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = \lim_{x\to\infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1. \tag{1}$$

Although Tschebyschef got tantalizingly close to proving (1) around 1850, the actual proof had to wait until Riemann introduced his zeta function in 1859. The locations of the zeros of the Riemann zeta function are related intimately to the distribution of primes. In particular, the statement that there are no zeros with real part 1 is equivalent to the Prime Number Theorem (1). In fact, the proof of the former statement by Hadamard and Poussin independently in

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1896 resulted in the proof of the Prime Number Theorem, a century after its statement [4, 7]. However, the connection between the Riemann zeta function and primes extends further. Conjectured by Riemann in his 1859 article, the Riemann hypothesis states that all (nontrivial) zeros of the Riemann zeta function have real part 1/2 [9]. Amazingly, this is actually equivalent to saying that the error in the claim $\pi(x) \sim \mathrm{Li}(x)$, is of order $\sqrt{x}\log x$ [3]. Here, we provide the definition and some properties of the Riemann zeta function and discuss its zeros and their deep connection with the primes.

2 The Dirichlet Series and Euler's Product

To understand the Riemann hypothesis more fully, we must define the Riemann zeta function. The first of many steps is study of the Dirichlet series in a complex variable $s = \sigma + \imath t$, which is

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

Like its real counterpart, the Dirichlet series diverges when $\Re(s) \leq 1$ and converges for $\Re(s) > 1$ [3]. A very special property of the Dirichlet series is its relation to the primes. This should be expected, as it is a sum involving natural numbers, each of which has a unique prime representation. In finding this relation, the hardest step is noting that,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots). \tag{2}$$

The product on the right is over all primes. Without addressing convergence we can make a plausibility argument for this equality. Notice that given any natural number, it would be possible to find its inverse if the right side were expanded. Every power of every prime is present on the right, and after expanding the product, every power of every prime will meet every power of every other prime in every combination. Since every natural number has a unique representation as a product of primes, all the inverses of the natural numbers appear exactly once on the right. Moreover, every term in the expansion of the right side is the inverse of a natural number. The next simplifying step is painless. Remembering the convergence of the ever important infinite geometric series, equation (2) becomes,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}. \tag{3}$$

This remarkable relation is known as the Euler product formula [3, 4, 10]. It gains extra notoriety for being the definition of the Riemann zeta function $\zeta(s)$ for $\Re(s) > 1$ [2, 3]. We can immediately deduce some properties of the zeta function from this definition. Recognize that $\zeta(s)$ for $\Re(s) > 1$ converges by the convergence of the sum on the left, and never equals zero, as the product on the right never has a vanishing factor [3].

3 Analytic Continuation

The major problem with the Riemann zeta function as defined so far is the domain restriction of $\Re(s) > 1$. It would be particularly nice if it were defined everywhere in the complex plane. Fortunately, this is possible for analytic functions, of which $\zeta(s)$ as defined by (3) over the domain $\Re(s) > 1$ is an example. An analytic function f over a domain A in the complex plane has a uniquely valued derivative at every point in A; analyticity is the complex analogue of differentiability (on an open set) in real analysis. More to the point, the theorem in complex analysis that makes the continuation we need possible is that of analytic continuation: "...given functions f_1, analytic on domain D_1, and f_2, analytic on domain D_2, such that $D_1 \cap D_2 \neq \emptyset$ and $f_1 = f_2$ on $D_1 \cap D_2$, then $f_1 = f_2$ on $D_1 \cup D_2$" [3]. Note that this implies the analytic continuation f_2 of f_1 onto D_2 is unique. One must be careful of singular points during this process, but with only isolated singular points to contend with, this theorem holds [1]. The uniqueness part of this theorem expresses clearly how stringent the requirement of analyticity is; after all, there is nothing as powerful as analytic continuation for functions of real numbers.

Thus, we can analytically continue $\zeta(s)$ over the entire complex domain except for the singularity at $s = 1$ if we find a function that agrees with the Dirichlet series on the domain $\Re(s) > 1$ and is analytic everywhere in $\mathbb{C}$. We follow Riemann's development [9] as clearly communicated by Edwards [5]. In order to find the continuation of the zeta function, Riemann started with Euler's gamma function integral. As a brief introduction, the gamma function extends the factorial function to non-integral values, in the sense that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ [3]. Euler's integral representation of this function, convergent for a complex number, $s = \sigma + \imath t$, with positive real part, is

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$

If $s = n \in \mathbb{N}$, it is easy enough to show that this formula gives $(n-1)!$ through integration by parts. Riemann makes the clever change of variables $x \to n^2 \pi x$ in $\Gamma(s/2)$ to find,

$$\Gamma(\frac{s}{2})\pi^{-s/2}\frac{1}{n^s} = \int_0^\infty x^{s/2-1} e^{-n^2\pi x} dx.$$

It really is a clever substitution because now we can sum n on both sides over all $\mathbb{N}$ to find, (interchanging integration and summation here is justified, but we omit the proof)

$$\Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s) = \int_0^\infty x^{s/2-1} \sum_{n=1}^{\infty} e^{-n^2\pi x} dx = \int_0^\infty x^{s/2-1} \left(\frac{\vartheta(x)-1}{2}\right) dx. \quad (4)$$

and, lo-and-behold, the zeta function as defined by (3) appears on the left side. Also, Riemann knew of the function that appears on the right by way of Jacobi, who studied

$$\vartheta(x) := \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x},$$

which converges for all $x > 0$ and satisfies an interesting equation that we make no attempt to prove:

$$\frac{\vartheta(x)}{\vartheta(1/x)} = \frac{1}{\sqrt{x}}. \tag{5}$$

Make note of the exponential speed with which $\vartheta(x) \to 0$ as $x \to \infty$ [5, 9].

Let's just consider the integral on the right of (4) for now. Notice that $\vartheta(x)$ is ill-behaved at $x = 0$; we need to pay special attention to the integrand near zero. Split the integral apart at $x = 1$ and perform the change of variables $x \to 1/x$ (so $dx \to -dx/x^2$) on the piece integrating from 0 to 1, getting,

$$\begin{aligned} \chi(s) := \int_0^\infty x^{s/2-1}\left(\frac{\vartheta(x)-1}{2}\right)dx = \int_1^\infty x^{s/2-1}\left(\frac{\vartheta(x)-1}{2}\right)dx \\ + \int_1^\infty x^{-s/2-1}\left(\frac{\vartheta(1/x)-1}{2}\right)dx. \end{aligned} \tag{6}$$

We defined $\chi(s)$ simply for easy reference. Equation (6) looks good; not only do the limits of integration match but also we can use (5) to find,

$$\frac{\vartheta(1/x)-1}{2} = \frac{\sqrt{x}\ \vartheta(x)-1}{2} = \sqrt{x}\left(\frac{\vartheta(x)-1}{2}\right) + \frac{\sqrt{x}-1}{2},$$

and then it follows that from (6),

$$\begin{aligned} \chi(s) = \int_1^\infty (x^{s/2-1} + x^{(1-s)/2-1})\left(\frac{\vartheta(x)-1}{2}\right)dx \\ + \int_1^\infty \frac{1}{2}\left[x^{(1-s)/2-1} - x^{-s/2-1}\right]dx. \end{aligned}$$

Finally, an integral that is easily integrable. For $\Re(s) > 1$, the second integral converges. We get

$$\int_1^\infty \frac{1}{2}\left[x^{(1-s)/2-1} - x^{-s/2-1}\right]dx = \left[\frac{x^{(1-s)/2}}{1-s} + \frac{x^{-s/2}}{s}\right]_1^\infty = \frac{-1}{s(1-s)},$$

so that,

$$\begin{aligned} \chi(s) &= \Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s) \\ &= \frac{-1}{s(1-s)} + \int_1^\infty (x^{s/2-1} + x^{(1-s)/2-1})\left(\frac{\vartheta(x)-1}{2}\right)dx, \end{aligned} \tag{7}$$

for $\Re(s) > 1$ [5, 9].

Did we really achieve anything? According to the derivation, it is just another equation valid for $\Re(s) > 1$, but notice by the exponential convergence

$$\lim_{x\to\infty}\left(\frac{\vartheta(x)-1}{2}\right) = \lim_{x\to\infty}\left(\sum_{n=1}^{\infty} e^{-n^2\pi x}\right) = 0$$

the integral on the right of (7) actually converges for all $s \in \mathbb{C}$ [5]. Also, we can use the analytic continuation of $1/\Gamma(s)$ to the whole complex plane, Weierstrass' formula,

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-s/n}, \tag{8}$$

where γ is the Euler-Mascheroni constant ($\gamma = 0.57721566...$) [3, 8].

Thus, equation (7) is exactly what we were looking for, an equation involving $\zeta(s)$ that is convergent (mostly) everywhere in the complex plane. This is the analytic continuation of $\zeta(s)$ as defined in (3) [3, 5, 9]:

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\left[\frac{-1}{s(1-s)} + \int_1^{\infty}(x^{s/2-1} + x^{(1-s)/2-1})(\frac{\vartheta(x)-1}{2})dx\right]. \tag{9}$$

There are other continuations of $\zeta(s)$ [5, 8, 9]; however, by the uniqueness of analytic continuations, they all equal equation (9) in whatever domain they are defined.

4 Properties of the Riemann Zeta Function

Note three important properties of $\zeta(s)$ following from the continuation (9). Though we will not prove it, the first is that $\zeta(s)$ is analytic at all points in $\mathbb{C}$ except for the simple pole at $s = 1$, which corresponds to the harmonic series. The second property arises from the well-known simple (degree one) poles of $\Gamma(s)$ at $s = 0, -1, -2, \cdots$, which are easily seen by consideration of (8). Thus, $\zeta(s)$ has so-called trivial zeros at $s = -2, -4, -6, \cdots$. The singularity $\Gamma(0)$ is canceled by the singularity of the bracketed factor in (9) at $s = 0$ since both are degree one poles. Finally, as a general rule, always make sure to note symmetry. In this case, [2, 3]

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})\zeta(1-s). \tag{10}$$

In words, the factor in brackets in (9) is symmetric around the point $s = 1/2$. To elevate this symmetry's importance, we define $\xi(s)$ to be the symmetric part of $\zeta(s)$ times the inherently symmetric factor $-s(1-s)/2$, [3]

$$\xi(s) := \frac{-s(1-s)}{2}\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) \tag{11}$$

$$= \frac{1}{2} - \frac{s(1-s)}{2}\int_1^{\infty}(x^{s/2-1} + x^{(1-s)/2-1})(\frac{\vartheta(x)-1}{2})dx. \tag{12}$$

Combination of (10) and (11) reveal the simple functional equation, [3]

$$\xi(s) = \xi(1-s). \tag{13}$$

This symmetry immediately yields pay dirt. By working with the product $\Gamma(s/2)\zeta(s)$, we have taken the trivial zeros of $\zeta(s)$ created by the poles of $\Gamma(s/2)$ into account. A glance at (8), in its convenient product form, shows $\Gamma(s/2)$ is never 0. Moreover, what would have been zeros of $\xi(s)$ at $s = 0, 1$ actually cancel with the poles $\Gamma(0)$ and $\zeta(1)$ respectively. We conclude that all nontrivial zeros of $\zeta(s)$ correspond exactly to the zeros of $\xi(s)$. Furthermore, notice that in the domain $\Re(s) > 1$, $\xi(s)$ has no zeros, since in that area

$$\xi(s) = -\frac{s(1-s)}{2}\pi^{-s/2}\Gamma(\frac{s}{2})\prod_p(1-\frac{1}{p^s})^{-1},$$

and $\Gamma(s/2)$, $s(s-1)$, and the Euler product have no zeros for $\Re(s) > 1$. Thus, by the invaluable symmetry (10), $\xi(s)$ has no zeros for $\Re(s) < 0$ and so $\zeta(s)$ has no nontrivial zeros for $\Re(s) < 0$. In other words, all the nontrivial zeros of $\zeta(s)$ lie in the strip $0 \leq \Re(s) \leq 1$. This region is called the *critical strip* and the line $\Re(s) = 1/2$ is termed the *critical line* [3].

There is another symmetry of $\xi(s)$ we should illustrate, namely,

$$\overline{\xi(s)} = \xi(\overline{s}). \tag{14}$$

The symmetry (14) follows from the definition of $\xi(s)$, which involves no outstanding $\imath$'s. All $\imath$'s in $\xi(s)$ are contained in s, so when taking the complex conjugate of $\xi(s)$, it suffices to take the complex conjugate of s and plug that into ξ. The important consequence of this symmetry is that if $s = s_0$ is a zero of $\xi(s)$, $\xi(s_0) = 0 = \overline{0} = \overline{\xi(s_0)} = \xi(\overline{s_0})$. So we see that $s = \overline{s_0}$ is a zero of $\xi(s)$ as well. Through application of the symmetry (13), we see $s = 1 - s_0$ and $s = 1 - \overline{s_0}$ are also zeros of $\xi(s)$. The same symmetries hold true for the nontrivial zeros of $\zeta(s)$ by their correspondence with the zeros of $\xi(s)$. It helps to imagine the geometry. These symmetries, (13) and (14), express the fact that the nontrivial zeros of $\zeta(s)$ are symmetric about both the critical line $\Re(s) = 1/2$ and the real axis $\Im(s) = 0$ [3].

As a final property of the ξ function, we show that $\xi(s) \in \mathbb{R}$, if $\Re(s) = 1/2$. This follows from equation (12). The factor multiplying the integral in (12) with $\Re(s) = \sigma = 1/2$ is

$$\frac{s(1-s)}{2} = \frac{(1/2+\imath t)(1/2-\imath t)}{2} = \frac{1/4+t^2}{2}.$$

It is obviously real for all $t \in \mathbb{R}$. Likewise, the integrand with $\sigma = 1/2$ is

$$\begin{aligned}(x^{s/2-1} + x^{(1-s)/2-1})(\frac{\vartheta(x)-1}{2}) &= (x^{-3/4}x^{\imath t/2} + x^{-3/4}x^{-\imath t/2})(\frac{\vartheta(x)-1}{2}) \\ &= x^{-3/4}\cos\big(\ln(x)\frac{t}{2}\big)(\vartheta(x)-1), \end{aligned} \tag{15}$$

where we used one of Euler's many famous formulas: $e^{\imath\theta} = \cos(\theta) + \imath\sin(\theta)$. The integral traverses the interval $x \in (1, \infty)$ where (15) stays real throughout. Therefore, $\xi(1/2 + \imath t)$ is real for all $t \in \mathbb{R}$. To introduce other frequently used notation, if t is allowed to be complex [5, 9],

$$\Xi(t) := \xi(\frac{1}{2} + \imath t).$$

5 The Zeros and the Primes

We found above that all nontrivial zeros of the Riemann zeta function lie in the critical strip, $0 \leq \Re(s) \leq 1$. This is, to say the least, the easy part of locating the zeros of $\zeta(s)$. The currently unresolved Riemann hypothesis states that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$ or, equivalently, as originally stated by Riemann, all the zeros of $\Xi(t)$ are real [3, 9]. In the introduction, we said that the Riemann Hypothesis is equivalent to the error in the prime number theorem (1) being of order $\sqrt{x}\ln x$ [2, 3, 4]. This is true and interesting, but connecting the zeros of the Riemann zeta function to the prime counting function $\pi(x)$ is where the real magic happens.

Amazingly, Riemann's hypothesis expresses the error in the prime number theorem (1) as the sum of waves with wavelength determined by the locations of the zeros of $\zeta(s)$. In an approximate sense, for ease of understanding, if the Riemann hypothesis is true,

$$\frac{\text{Li}(x) - \pi(x)}{\sqrt{x}/\ln x} \approx 1 + 2\sum_{\gamma \in Y} \frac{\sin(\gamma \ln x)}{\gamma}, \tag{16}$$

where $Y = \{\gamma \in \mathbb{R}^+ : \Xi(\gamma) = 0\}$. The zeros of the Riemann zeta function are placed ever so precisely that they anticipate the locations of the primes! Moreover, they do it through the Fourier series type term in (16) in which the values of the zeros determine both the amplitude and the wavelength-analog of the sine-log waves [6]. This is exactly what was meant by Berry and Keating when they said the Riemann hypothesis is equivalent to the statement, "The primes have music in them"[3]. If the Riemann hypothesis is false, this formula would be marred with more complicated coefficients for the sines that are functions of x [6].

The nontrivial zeros of the Riemann zeta function are obviously incredibly important, so we would be amiss if we did not at least briefly communicate a few of their properties. First of all, there are an infinite number of them on the critical line; the sum in (16) is infinite. This fact was proven by G. H. Hardy in 1914. Secondly, the number of zeros in a rectangle $0 < \sigma < 1$, $0 \leq t < H$, denoted $N(H)$, is

$$N(H) = \frac{H}{2\pi}\left[\ln\frac{H}{2\pi} - 1\right] + O(\ln H). \tag{17}$$

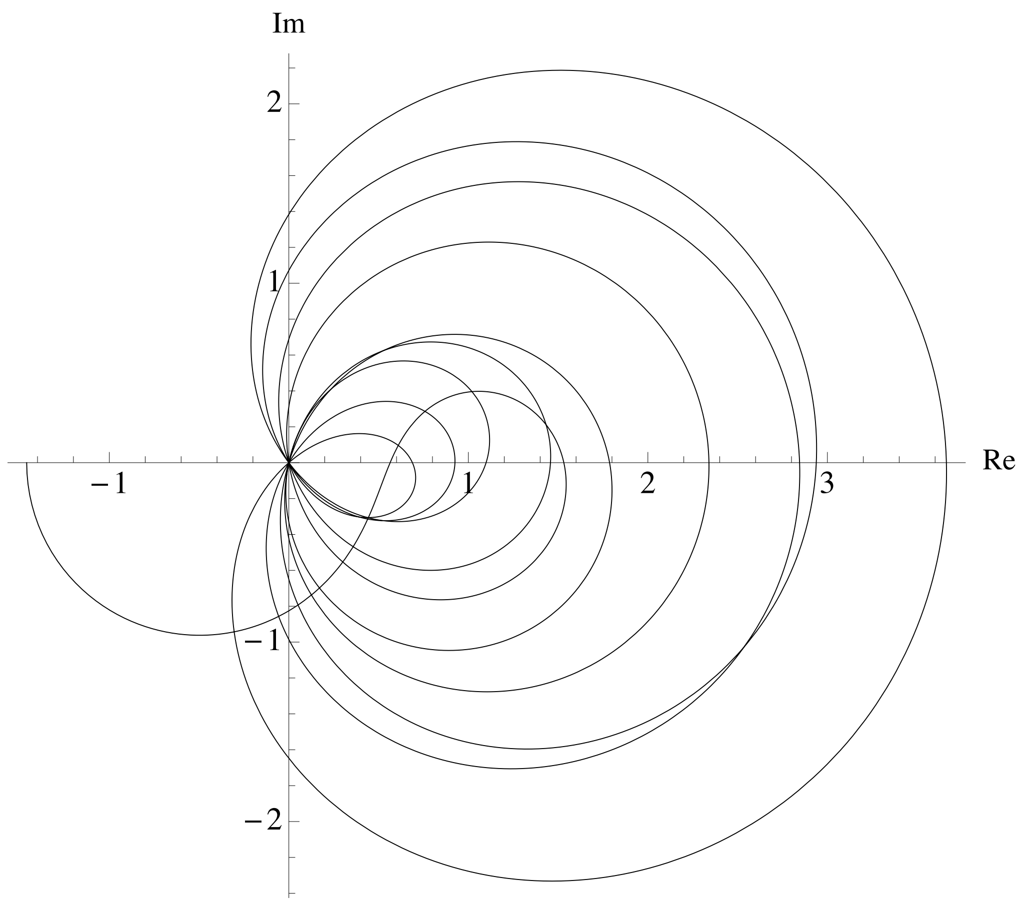

Figure 1: Riemann's zeta function on the critical line $1/2 + \imath t$ is graphed parametrically with parameter t from zero up to the tenth positive zero. The first ten values of t for which $\zeta(1/2 + \imath t) = 0$ are (to the hundredths places) 14.13, 21.02, 25.01, 30.42, 32.94, 37.59, 40.92, 43.33, 48.01, and 49.77. This image and the values of the zeros were calculated using Mathematica code [11].

This formula was mentioned along with a non-rigorous argument in the same landmark paper where Riemann proposed his hypothesis. However, (17) turned out to be an easier nut to crack for it was proven in 1905 by von Mangoldt [3, 5].

The placement of the zeros of ζ along the critical line is also an interesting topic. One can talk about the statistics of the gaps between zeros on this line. And so, as a final property of the zeros of $\zeta(s)$, we state Hugh L. Montgomery's conjecture from 1973: along the critical line, the expected number of zeros following a zero in an interval of length $T > 0$ times the average gap is [6]

$$M(T) = \int_0^T 1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2 du = T - \int_0^T \left(\frac{\sin(\pi u)}{\pi u}\right)^2 du.$$

Montgomery conjectured this relationship based upon numerical data. Note that if the zeros were distributed randomly, $M(T)$ should simply be T. We see that Montgomery's conjecture implies $M(T) < T$ since $\int_0^T \left(\frac{\sin(\pi u)}{\pi u}\right)^2 du > 0$ for all $T > 0$. Thus, the nontrivial zeros of the zeta function are more widely spaced than a pure random distribution; there are fewer zeros just after a zero than a random distribution would predict. Sometimes this property is referred to as repulsion between the zeros of $\zeta(s)$. What is remarkable is that the same distribution appears in the study of the spacings of energy levels of quantum chaotic systems [6]. The precise language is that "The distribution of spacings between nontrivial zeros of the Riemann zeta function is statistically identical to the distribution of eigenvalue spacings in a Gaussian unitary ensemble" [3]. This connection with the relatively simple mathematics of quantum chaos is a promising direction for a proof of the Riemann hypothesis.

6 Conclusion

In summary, the Riemann hypothesis is arguably the most important unsolved problem in contemporary mathematics due to its deep relation to the fundamental building blocks of the integers, the primes. Also, in the sense of (16) its truth would guarantee the nicest possible distribution of the primes. That appeal to beauty is the basis of many mathematicians confidence in the Riemann hypothesis. Admittedly, the computational endeavor that has found around 10 trillion zeros on the critical line and none off it as of 2004 does bolster faith as well [3]. As a consequence of the widely held belief in its truth, many results in number theory are proven *assuming* the Riemann hypothesis. Many others are found to be equivalent to the Riemann hypothesis. The proof of the Riemann hypothesis will immediately verify a slew of dependent theorems [3, 10]. For these many reasons, when the great mathematician David Hilbert was asked what he would do if he were to be revived in five hundred years, he replied, "I would ask, 'Has somebody proven the Riemann hypothesis?'"[10] Hopefully, by that time, the answer will be, "Yes, of course."

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