

MATHEMATICAL MODELING OF CLIMATE CHANGE
EXAMINING THE DIURNAL TEMPERATURE RANGE

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Abstract

In this paper, I work my way through Robert M. Gethner's article entitled *Climate Change and the Daily Temperature Cycle*. I examine his climate model closely and take a deeper look at the mathematics that he uses to develop his model. The model I will look at works towards a better understanding of both the average global surface temperature as well as the daily fluctuations in temperature. I determine that on a planet without an atmosphere, the temperatures are much colder than on planets with atmospheres and I discuss how the presence of an atmosphere affects global temperatures. I look at how changes in the atmospheric composition have an effect on global temperatures as well. Finally, I look at how changes in the average global surface temperature affect changes in the daily temperature cycle. I conclude that, at the moment, the Earth's surface temperature is rising, while daily fluctuations in temperature have been decreasing.

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CHAPTER 1

Introduction

In this paper I will examine the mathematical model discussed by Robert M. Gethner in his paper, *Climate Change and the Daily Temperature Cycle*[4]. His model can be used to make useful predictions and calculations with regard to global climate change. Global climate change, commonly referred to as ‘global warming,’ is a very complex process in which global climate trends have changed through time, especially in recent years, largely due to the impact of human industrialization and population growth[2]. Because climate change is such a vast and complex field of study, I will focus specifically on one aspect of climate change: the daily temperature cycle. In general, daily fluctuations in temperature are attributed to weather, not climate. Climate implies a long term trend, whereas weather accounts for daily and seasonal variations. That being said, the particular changes in the daily temperature cycle that I plan to investigate further represent a long term trend in daily temperature changes, which would make my model a climate model, not a weather model. I will look at how surface temperature has changed and is changing through time. I also plan to investigate how the *diurnal temperature range* (DTR) has changed, and will continue to change, through time.

DEFINITION 1. The *diurnal temperature range* is the difference between the minimum temperature (at night) and the maximum temperature (during the day).

My model will show that the average surface temperature is increasing through time, but that the DTR is actually decreasing as a result of climate change. This means that, even though the average global temperatures have been increasing, the difference between daytime highs and night time lows will decrease. This will most likely manifest itself in warmer nights, which can have a severe impact on plant growth and therefore it will lead to an indirect effect on various organisms, including humans.

In reading this paper, it will be important to understand how to develop and interpret mathematical models of climate change. First of all, as with any model, many assumptions will be made. This does not mean the results of the model are useless and unrealistic. What this means is that, in interpreting the results of the model, one must keep in mind the assumptions that were made. Even though the model will not be perfectly accurate, it can still provide us with useful information and, perhaps, lead us to ask questions for future research. It is also important to note that, while many assumptions will be made towards the beginning of the modeling process, we will work to gradually break down each assumption one by one, although some will remain. At the beginning of this paper, we will start with a generic planet that is similar to Earth, but it will have no atmosphere, no oceans, and it will be at a uniform temperature (Chapter 3). We will eventually introduce an atmosphere into our model, which will make a substantial difference in the complexity and usefulness of the model (Chapter 6). Once we introduce the atmosphere into the model, we will be able to see how anthropogenic effects such as emission of greenhouse gasses will effect the DTR (Chapter 6).

CHAPTER 2

Background

Before we will be able to develop a meaningful model for climate, we need to discuss some important concepts regarding climate, power, and basic planetary physics. A good way to start this discussion will be to talk about units.

2.1. Units

In the scientific world, units are very important to help us understand what numerical values assigned to various things actually mean. For instance, 1 foot means something very different than 1 mile. It is also important to know what sort of thing different units measure. A mile is a measure of distance, but a pound is a measure of weight. When talking about climate, it is important to understand units of energy, power, and temperature.

Energy is usually measured in Joules (J), but can also be measured in British Thermal Units (Btus), Kilowatt-Hours (kWh), or Newton-Meters (Nm) among others. Power is actually a rate and its units are always of the form Energy/unit time. A common unit for power is the watt (W), which is equivalent to 1 J/s. Another unit for power would be Btus/minute. Temperature can be measured in degrees Fahrenheit ($^{\circ}\text{F}$), degrees Celsius ($^{\circ}\text{C}$), or in Kelvin (K). The freezing point of water is 32°F , 0°C , and 273 K, to give a basic sense of the scale of each unit. In this paper, we will mostly use Joules to describe amounts of energy, watts to describe amounts of power, and Kelvins to describe temperatures. There will be other things that we will talk about that will involve these things in various ways and use related units. For instance, we will talk about *heat capacity*.

DEFINITION 2. *Heat capacity* is the amount of energy it takes to raise the temperature of an object by 1 Kelvin. We will denote this as C .

As you may be able to guess, the units used to describe different heat capacities are J/K.

2.2. Solar Flux

When discussing the Earth's climate, it is important to have a good understanding of where the power that is used to heat the Earth comes from. We refer to the incoming power from the sun as the Solar Flux. It can be calculated using a number that differs for each planet called the solar constant, which we will denote as Ω . Ω represents the total power that reaches the planet for every square meter of surface area on the planet[4]. For the Earth, $\Omega = 1372 \text{ W/m}^2$ [4]. Notice that the units for the solar constant are in the form of power/unit area. In other words, the solar constant represents the power from the sun that is cast upon each square meter of the Earth's surface.

It should be noted that at any given time, the sun is only shining on one side of the Earth. From the sun's perspective, the Earth is just a circle, not a sphere, so Solar Flux is received by a circle with area πr^2 where r is the radius of the Earth. If we average the solar power received by that disk ($\pi r^2 \Omega$) over the entire surface area of the Earth ($4\pi r^2$), we get that the average solar power received by the Earth per unit area of the Earth's surface is $\pi r^2 \Omega / 4\pi r^2 = \Omega/4$. This is still an overestimate for the actual power received by each square meter of the Earth's surface because it does not take the Earth's *albedo* into account.

DEFINITION 3. A planet's *albedo* is the percent of incoming solar radiation that is immediately reflected back into space due to the coloring of the planet. A completely white planet would have an albedo of 1 and a completely black planet would have an albedo of 0.

The Earth's albedo, denoted α , is 0.3, so 30% of incoming solar radiation is immediately reflected back into space[4]. This means that the fraction $a = 1 - \alpha$, the *co-albedo*, of incoming solar radiation actually makes it to the planet's surface. This shows us that the total amount of incoming solar radiation, Flux In, is equal to $a\Omega/4$.

2.3. Flux Out

It is clear at this point that if there is a Flux In, there must be a Flux Out. Otherwise the Earth would continue to heat up towards infinity, which is clearly not the case. In order to determine a formula for the Flux Out, we must first make a

rather large assumption about the Earth. We must assume that the Earth behaves like a *blackbody*.

DEFINITION 4. A *blackbody* is an object that fits the following criteria:

- (1) It is at a uniform temperature (no internal temperature variations).
- (2) All of the object's intercepted radiation is absorbed by the object. (i.e. No radiation is reflected.)
- (3) It emits radiation at its maximum rate.

Given this assumption, we can use the *Stefan-Boltzmann Law* to write an expression for Flux Out.

DEFINITION 5. The *Stefan-Boltzmann Law* states that, for a blackbody:

$$(1) \quad \text{Flux Out} = \sigma T^4$$

where σ is the *Stefan-Boltzmann constant* ($\sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$) and T is the temperature of the blackbody.

2.4. Effective Temperature

DEFINITION 6. The *effective temperature* of a planet, T_e , is what it's temperature would be given the following conditions:

- (1) It is at a uniform temperature.
- (2) It absorbs the same amount of power that it emits.
- (3) The rate at which it emits power is equivalent to the rate at which a blackbody at the same temperature would emit power.

By the definition of T_e , we can see that Flux In = Flux Out where Flux In = $a\Omega/4$ and Flux Out = σT_e^4 because for the effective temperature, the planet is behaving like a blackbody. We can use this information to derive a formula for a planet's effective temperature:

$$(2) \quad \begin{aligned} a\Omega/4 &= \sigma T_e^4 \\ \Rightarrow T_e &= (a\Omega/4\sigma)^{1/4} \end{aligned}$$

At this point one might ask why it is important to calculate a planet's effective temperature. As we build our climate model, we will begin with a planet with no atmosphere. We will later add an atmosphere to make our model more realistic,

but it is important that we first understand how a planet's climate works when there is no atmosphere. That being said, it turns out that for a planet without an atmosphere, the effective temperature is equivalent to the surface temperature of the planet, so it will be important that we know how to calculate the effective temperature. Of course, in reality, the Earth's average surface temperature is about 35 K warmer than its effective temperature[4].

2.5. Radiation

When we talk about Solar Flux, we are really talking about solar radiation.

DEFINITION 7. Radiation is a form of heat transfer resulting from the random thermal motion of atoms and molecules[8].

Depending on the speed of the movement of the atoms in a given object, radiation occurs at different wavelengths. Visible light, i.e. the colors that we see in our world, is the result of the reflection of the sun's radiation off the objects that we see around us. The sun's radiation generally falls within a range of short wavelengths (0.15-4.0 μm), but the visible light (between 0.4 and 0.8 μm) only accounts for 49% of solar radiation[5]. Wavelengths less than 0.4 μm are referred to as ultraviolet radiation and wavelengths near the other end of the visible range are referred to as infrared radiation. Compared to the entirety of the electromagnetic radiation spectrum, the range from ultraviolet to infrared radiation is actually quite small, but it will be the only part of that spectrum that we will focus on in this paper because we are only concerned with solar radiation, which exists within that range, and with the re-radiation of energy from the Earth (discussed above in Section 2.3), which generally occurs within the infrared range. We will refer to the ultraviolet and visible ranges as 'shortwave radiation' and we will refer to the infrared range as 'longwave radiation.'

CHAPTER 3

Initial Temperature Calculations: No Atmosphere

3.1. Setting Up the Equations

In this section, we discuss a planet with no atmosphere. Because there is no atmosphere, it is safe to say that

$$(3) \quad \hat{T} = T_e,$$

where \hat{T} is the planet's equilibrium surface temperature, $a\Omega/4 = \sigma\hat{T}^4$ where $a\Omega/4 = \text{Flux In}$ and $\sigma\hat{T}^4 = \text{Flux Out}$, and T_e is the planet's effective temperature. The effective temperature is the temperature when Power Emitted = Power Absorbed and power is emitted at the same rate that, at the same temperature, a blackbody would emit power.

We can use equation (2) to determine the effect that a change in the solar constant (Ω) will have on the effective temperature of our planet (T_e). We will start by taking the derivative of T_e with respect to Ω and then we will isolate ΔT_e , the change in the effective temperature.

$$\begin{aligned} \frac{dT_e}{d\Omega} &= \left(\frac{a}{4\sigma}\right)^{1/4} \cdot \frac{1}{4}\Omega^{-3/4} \\ &= \left(\frac{a}{4\sigma}\right)^{1/4} \cdot \frac{1}{4} \cdot \left(\frac{1}{\Omega^3}\right)^{1/4} \\ &= \frac{1}{4} \left(\frac{a}{4\sigma\Omega^3}\right)^{1/4} \\ &= \left(\frac{1}{256}\right)^{1/4} \left(\frac{a}{4\sigma\Omega^3}\right)^{1/4} \\ &= \left(\frac{a}{1024\sigma\Omega^3}\right)^{1/4}, \end{aligned}$$

so we have:

$$(4) \quad \Delta T_e = \frac{dT_e}{d\Omega} \Delta\Omega = \left(\frac{a}{1024\sigma\Omega^3} \right)^{1/4} \Delta\Omega.$$

Given this model, we can discuss what would happen if the solar constant increased suddenly, reaching a new (constant) value. This sudden change would cause the planet's surface to absorb more power than it emits. Therefore the planet will no longer be in equilibrium leading to a rise in the planet's net energy and temperature, T . With higher temperatures, our planet will emit energy at a faster rate. This will cause the temperature to rise, but at an increasingly slower rate. Hence T will approach a new equilibrium. For example, we can calculate how much the temperature on Mars would increase if the Martian solar constant were to increase by 1%.

CLAIM 8. *Equation (4) predicts that the temperature on Mars would increase by 0.54 K given a 1% increase in the Martian solar constant.*

PROOF. We know that $a_{mars} = 1 - 0.15 = 0.85$ [4] and the mean distance from Mars to the Sun = 227.94×10^6 km[4]. If we envision a sphere with the Sun at its center and with radius equal to the distance between the Earth and the Sun, we can think of Ω_{earth} as the power received by each square meter of that sphere. So by multiplying by the surface area of that sphere, $4\pi r^2$, we get the total power received by the sphere:

$$4\pi(149.6 \times 10^9 \text{ m})^2(1372 \text{ W/m}^2) = 3.86 \times 10^{26} \text{ W}.$$

This same amount of power will also be received by a sphere with radius equal to the distance between Mars and the Sun, but will be spread out over a larger area[4].

Therefore we can determine that $\Omega_{mars} = \frac{3.86 \times 10^{26} \text{ W}}{4\pi(227.94 \times 10^9 \text{ m})^2} = 591 \text{ W/m}^2$.

Equation (4) implies that

$$\Delta T_e = \left(\frac{0.85}{1024(5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4)(591 \text{ W/m}^2)^3} \right)^{1/4} (0.01)(591 \text{ W/m}^2),$$

which is equal to 0.54 K □

In this scenario, the temperature asymptotically approaches the new equilibrium, so the temperature never actually reaches the new equilibrium value. It now

makes sense to ask how long it would take for the temperature to get, say, 95% closer to the new equilibrium from its value when the solar constant is changed.

To do this, we first want to derive a differential equation for T such that both Flux In = $a\Omega/4$ and Flux Out = σT^4 do not necessarily have to hold. Since flux is power per unit area (emitted or absorbed), we can calculate the amount of energy that the planet's surface will absorb during a short time interval $[t, t + \Delta t]$ to be about (Power In) Δt where Power In represents the power at time t . (Power In) Δt is merely an approximation of the energy absorbed between time t and $t + \Delta t$ because Power In may change during that time. Since we're assuming that Δt is small, though, the change will probably not be significant. (Smaller values of Δt lead to more accurate approximations.) We can use similar logic to write the amount of energy that the planet's surface will emit during the same interval as (Power Out) Δt . We can combine these expressions to show that the net change in energy is (Power In - Power Out) Δt .

We can now make the assumption that since our planet has no atmosphere, all of the energy absorbed contributes to increasing the planet's temperature. With this assumption, we can write the net change in energy as $C \cdot \Delta T$, where C is the surface's heat capacity (Definition 2). We now have $C \cdot \Delta T = \text{net change in energy} \approx (\text{Power In} - \text{Power Out})\Delta t$, so $C \cdot \Delta T \approx (\text{Power In} - \text{Power Out})\Delta t$. In order to derive our differential equation for T , we divide both sides by Δt . We take the limit as Δt approaches, but does not equal, zero to make this division possible.

This gives us:

$$(5) \quad C \frac{dT}{dt} = \text{Power In} - \text{Power Out}.$$

$C \frac{dT}{dt}$ is called the heat storage term meaning the amount of energy added to the surface per unit time. C is in units of Joules/Kelvin, so when multiplied by $\frac{dT}{dt}$, which has units of Kelvin/unit time, the result is a term that has units Joules/unit time. We will now define a few new variables:

$$S = \text{the planet's surface area (m}^2\text{)},$$

$$c = C/S.$$

Thus, c is the heat capacity per square meter of the planet's surface.

Using these new variables, we will examine the case where, in fact, Flux In = $a\Omega/4$ and Flux Out = σT^4 .

This implies:

$$\begin{aligned}\text{Power In} &= (a\Omega/4) \cdot S = \frac{Sa\Omega}{4} \\ \text{Power Out} &= (\sigma T^4) \cdot S = S\sigma T^4.\end{aligned}$$

We also know that:

$$C = cS.$$

Plugging all of these values into (5), we get:

$$(6) \quad cS \frac{dT}{dt} = \frac{Sa\Omega}{4} - S\sigma T^4.$$

We can factor out and then cancel all of the S 's. We can also apply equation (2) to get a nonlinear differential equation for T :

$$(7) \quad c \frac{dT}{dt} = \sigma T_e^4 - \sigma T^4.$$

To make finding a solution easier, we will linearize the equation. We do this by letting u be the displacement of surface temperature from it's original equilibrium value, so $u = T - T_e$. We can now replace T in our differential equation by $T_e + u$. This gives us:

$$(8) \quad c \frac{d(T_e + u)}{dt} = \sigma T_e^4 - \sigma(T_e + u)^4.$$

Since $\frac{dT_e}{dt} = 0$ (effective temperature is constant through time), this becomes a differential equation of u , which can be simplified as follows:

$$\begin{aligned}c \frac{du}{dt} &= \sigma T_e^4 - \sigma(T_e + u)^4 \\ &= \sigma T_e^4 - \sigma(T_e^4 + 4T_e^3u + 6T_e^2u^2 + 4T_eu^3 + u^4) \\ &= \sigma T_e^4 - \sigma T_e^4 - 4\sigma T_e^3u - 6\sigma T_e^2u^2 - 4\sigma T_eu^3 - \sigma u^4.\end{aligned}$$

Because we are assuming that the difference between the temperature and the effective temperature, u , will be sufficiently small, we can disregard all terms in which u is raised to a power higher than one. After cancellation, this gives us:

$$(9) \quad c \frac{du}{dt} = -4\sigma T_e^3u.$$

We now have a homogeneous linear differential equation. Given that the displacement, u , isn't 'too large,' the solution of this equation should give us the displacement of surface temperature from the effective temperature T_e in terms of time. We will use this solution in Chapter 4 to answer the question; how long will it take in terms of t , time for u , the difference between T and T_e , to decrease by 95%?

3.2. A Preliminary Look at the Diurnal Temperature Range

We will now answer a different question: how large is the diurnal temperature range?

We will assume that our planet rotates with a period of P . Because we have no oceans or atmosphere, in order for energy to travel from one hemisphere (for instance the one experiencing day) to the other, it must conduct through the materials making up the surface. This process is sufficiently slow, so we can think of each hemisphere as isolated from the other. We will also assume that day and night each last for $P/2$ amount of time, i.e. day and night have equal lengths.

We continue to assume that Flux Out $= \sigma T^4$, but Flux In $\neq a\Omega/4$. Rather, throughout the day and night, the Flux In varies. We can assume, however, that over the course of a day, the average Flux In will be equal to $a\Omega/4$.

In order to derive a formula for Flux In over time, we must first define and discuss a few variables and functions:

DEFINITION 9. $\psi(s)$ is a piecewise continuous function, periodic of period 2 where

$$(10) \quad \int_0^2 \psi(s) ds = 0.$$

We can turn $\psi(s)$ into a function that is periodic of period P by setting $s = \frac{2t}{P}$. As s moves from 0 to 2, t moves from 0 to P . This follows since $s = 0 \Rightarrow t = 0$ and $s = 2 \Rightarrow t = P$. We can define the mean of a piecewise continuous function over the interval $[a, b]$ to be m such that:

$$(11) \quad \int_a^b f(x) dx = m(b-a),$$

$$\Rightarrow (b-a)^{-1} \int_a^b f(x) dx = m.$$

where m is the height of the rectangle formed by concentrating the area under f between a and b into a rectangular shape[6].

Since, in the case of the ψ function, $m = a\Omega/4$, let ψ represent the Flux In throughout the course of a day. We know that the mean Flux In will be $a\Omega/4$, so we can use equation (2) to show that Flux In should be centered around σT_e^4 . Our ψ function will thus determine how Flux In varies around σT_e^4 throughout the day. We can now write our function for Flux In(t) as:

$$(12) \quad \text{Flux In}(t) = \sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right].$$

If $\psi \left(\frac{2t}{P} \right) = 0$, then Flux In remains constant at $\sigma T_e^4 = a\Omega/4$. Because, as discussed earlier, $\psi \left(\frac{2t}{P} \right)$ is periodic with period P , the Flux In function is also periodic with period P .

Based on (11), we have, over the interval $[0, P]$, the mean value of Flux In is given by:

$$P^{-1} \int_0^P \sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] dt.$$

Substituting $s = \frac{2t}{P} \Rightarrow ds = \frac{2}{P} dt \Rightarrow \frac{P}{2} ds = dt$ we get:

$$(13) \quad \begin{aligned} \frac{1}{P} \int_0^2 \sigma T_e^4 [1 + \psi(s)] \frac{P}{2} ds &= \frac{1}{2} \sigma T_e^4 \int_0^2 [1 + \psi(s)] ds \\ &= \sigma T_e^4 + \sigma T_e^4 \int_0^2 \psi(s) ds. \end{aligned}$$

By (10), we have:

$$\sigma T_e^4 + \sigma T_e^4 \int_0^2 \psi(s) ds = \sigma T_e^4 = a\Omega/4,$$

which is what we wanted the mean value to be.

We can use this information, in conjunction with (5) to derive a differential equation of T as we did before:

$$(14) \quad \begin{aligned} cS \frac{dT}{dt} &= S\sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] - S\sigma T^4 \\ \Rightarrow c \frac{dT}{dt} &= \sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] - \sigma T^4. \end{aligned}$$

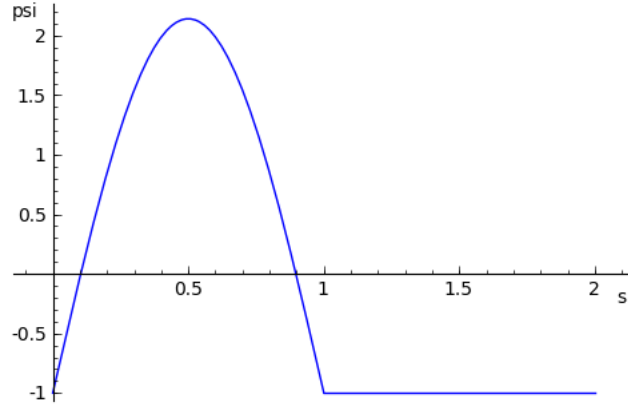
We will again linearize this equation by substituting $T_e + u$ for T :

$$\begin{aligned}
 (15) \quad c \frac{d(T_e + u)}{dt} &= \sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] - \sigma (T_e + u)^4 \\
 &\Rightarrow c \frac{du}{dt} = \sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] - \sigma T_e^4 - 4\sigma T_e^3 u \\
 &\Rightarrow c \frac{du}{dt} + 4\sigma T_e^3 u = \sigma T_e^4 \psi \left(\frac{2t}{P} \right).
 \end{aligned}$$

We now have a non-homogeneous linear differential equation. The solution to this equation should give us the displacement of surface temperature from the effective temperature T_e in terms of time given that the displacement, u , isn't 'too large' and the planet experiences both night and day.

One question we have yet to address is what we should use for ψ . We will discuss three possible options, although it should be noted that there are an infinite number of functions that satisfy the criteria for ψ .

FIGURE 3.1. $\psi(s)$



1) If we want the Flux In to be zero at night and then to gradually rise and fall throughout the day, one possibility would be (Figure 3.1):

$$(16) \quad \psi(s) = \begin{cases} \pi \sin(\pi s) - 1 & , 0 \leq s \leq 1 \\ -1 & , 1 \leq s \leq 2. \end{cases}$$

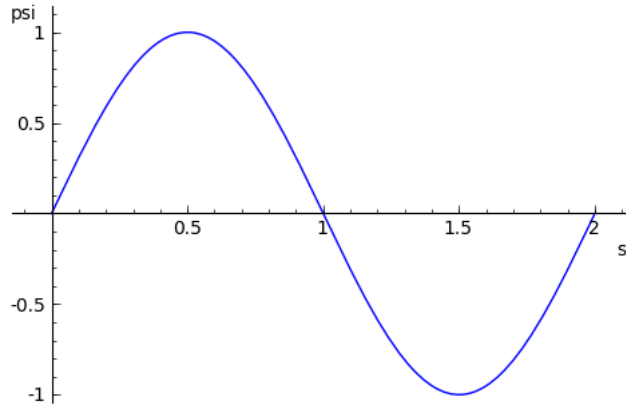
This formula would be repeated with period 2 (i.e. when $2 \leq s \leq 3$, $\psi(s) = \pi \sin(\pi s) - 1$ and when $3 \leq s \leq 4$, $\psi(s) = -1$, etc.) We must confirm that it

satisfies $\int_0^2 \psi(s)ds = 0$. We have:

$$\begin{aligned} \int_0^2 \psi(s)ds &= \int_0^1 (\pi \sin(\pi s) - 1)ds + \int_1^2 -1ds \\ &= \int_0^\pi (\pi \sin(t)) \frac{1}{\pi} dt + \int_0^1 -1ds + \int_1^2 -1ds \\ &= [-\cos(t)]_0^\pi + [-s]_0^1 + [-s]_1^2 \\ &= 1 + 1 - 1 + (-2 + 1) = 0. \end{aligned}$$

Although this option satisfies the necessary criteria and even makes intuitive sense, it does not provide a simple method for determining diurnal temperature range because it would require complicated calculations.

FIGURE 3.2. $\psi(s), H = 1$

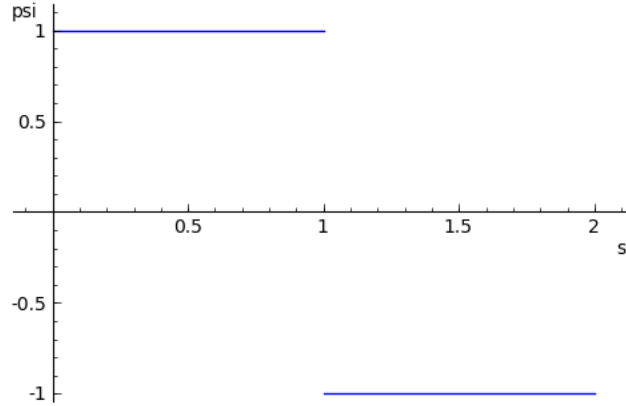


2) We will now look at a scenario where Flux In continuously rises and falls with period P (Figure 3.2). We can write this as:

$$(17) \quad \psi(s) = H \sin(\pi s),$$

where $0 \leq H \leq 1$. The following confirms that $\int_0^2 \psi(s)ds = 0$:

$$\begin{aligned} \int_0^2 \psi(s)ds &= \int_0^2 H \sin(\pi s)ds = H \int_0^2 \sin(\pi s)ds \\ &= \frac{H}{\pi} \int_0^{2\pi} \sin(t)dt = \frac{H}{\pi} [-\cos(t)]_0^{2\pi} \\ &= \frac{H}{\pi} [-1 - (-1)] = 0. \end{aligned}$$

FIGURE 3.3. $\psi(s)$ 

3) This option is much simpler and surprisingly works quite well. In this option, the sun is up during the day and down at night (Figure 3.3). It is written as:

$$(18) \quad \psi(s) = \begin{cases} 1 & , 0 \leq s \leq 1 \\ -1 & , 1 \leq s \leq 2. \end{cases}$$

Again, this formula is repeated with period 2.

Let's confirm that it fits the criteria for ψ :

$$\begin{aligned} \int_0^2 \psi(s) ds &= \int_0^1 1 ds + \int_1^2 -1 ds \\ &= [s]_0^1 + [-s]_1^2 \\ &= (1 - 0) + (-2 - (-1)) = 0. \end{aligned}$$

This leads to the following, reasonable calculation for the mean value of Flux In during the day (i.e. the mean value of Flux In during the interval $[0, \frac{P}{2}]$):

$$\left(\frac{P}{2}\right)^{-1} \int_0^{P/2} \sigma T_e^4 \left[1 + \psi\left(\frac{2t}{P}\right)\right] dt$$

substituting $s = \frac{2t}{P} \Rightarrow ds = \frac{2}{P}dt \Rightarrow \frac{P}{2}ds = dt$

$$(19) \quad \frac{2}{P} \int_0^1 \sigma T_e^4 [1 + \psi(s)] \frac{P}{2} ds = \sigma T_e^4 \int_0^1 [1 + \psi(s)] ds$$

$$(20) \quad \begin{aligned} &= \sigma T_e^4 (1 + \int_0^1 \psi(s) ds) \\ &= \sigma T_e^4 + \sigma T_e^4 = 2\sigma T_e^4 = 2a\Omega/4 \\ &= a\Omega/2. \end{aligned}$$

These three possibilities for ψ are just three of an infinite amount of choices one could make. In this paper, we will primarily be using equation (18) to represent ψ .

CHAPTER 4

Solving Some Important Differential Equations

We will now analyze equations (9) and (15) by first simplifying them. If we take $k = \frac{4\sigma T_e^3}{c}$ and $A = \frac{\sigma T_e^4}{c}$, we get the following differential equations:

$$(21) \quad \frac{du}{dt} + ku = 0$$

and

$$(22) \quad \frac{du}{dt} + ku = A\psi\left(\frac{2t}{P}\right).$$

We will start by finding a general solution for the homogeneous equation (21). Using separation of variables, we can infer that $u(t) = Ce^{-kt}$ where C is a constant. We can solve for C by letting $t = 0$. When we do this, we get $C = u(0)$. This gives us:

$$u(t) = u(0)e^{-kt}.$$

We can see that u is a decreasing function of t . Since we would like to measure how quickly u is reduced by 95% (i.e. how quickly the temperature approaches its equilibrium temperature), we will calculate the e-folding time of u .

DEFINITION 10. *e-folding time is the amount of time it takes for a function to decrease by a factor of e .*

PROPOSITION 11. *It will take time $3/k$ for the displacement in temperature to become 95% closer to 0.*

PROOF. We can solve for the e -folding time, denoted τ , by setting $u(t) = e \cdot u(t + \tau)$. This gives us:

$$\begin{aligned}
 (23) \quad e^{-1} &= \frac{u(t + \tau)}{u(t)} \\
 &= \frac{u(0)e^{-k(t+\tau)}}{u(0)e^{-kt}} \\
 &= \frac{e^{-kt-k\tau}}{e^{-kt}} \\
 &= e^{-k\tau} \\
 \Rightarrow \ln(e^{-1}) &= \ln(e^{-k\tau}) \\
 \Rightarrow -1 &= -k\tau \\
 \Rightarrow \tau &= \frac{1}{k}.
 \end{aligned}$$

Since the passing of time τ leads to a reduction in u by a factor of e , it is clear that the passing of time 3τ will lead to a reduction of u by a factor of $e^3 \approx 20$. Reducing u by a factor of 20 is equivalent to reducing it by 95% since $1/20 = 5/100$. Therefore, after the passing of time $3/k$, the displacement in temperature, $T - T_e$, will be 95% closer to 0. \square

We will now attempt to compute the amount of time it would take for the surface temperature to approach its new equilibrium value if the solar constant suddenly rose by 1%. We have already determined (Chapter 3) that the effective surface temperature, T_e , on Mars would rise by about 0.54 K if Mars' solar constant rose by 1%. Because the current T_e on Mars is 216.9 K[4], this would lead to a new T_e of 217.4 K

If we assume that the solar constant changed at time $t = 0$, then we know that $T(0) = 216.9$ K and the effective temperature, T_e , is equal to 217.4 K when $t > 0$. We can define the following function when $t > 0$: $u(t) = T(t) - T_e$. This means that when $t > 0$, $u(t) = T(t) - 217.4$ K. This is consistent with the definition of u in Chapter 3, so equation (9) is satisfied by u .

Recall that equation (23) states a formula for the e -folding time. Using this information, we can say that the e -folding time of u is:

$$(24) \quad \tau = \frac{c}{4\sigma T_e^3}.$$

In order to determine how long it will take for the surface temperature to approach the new equilibrium temperature, it will be useful to obtain a numerical value for τ . The challenge will be determining a numerical value for c . By definition, c is the surface's heat capacity per square meter of surface. It will prove useful to determine a value of c for Earth and then generalize it's use for Mars.

To calculate c , we need to determine the heat capacity of the materials that make up Earth's crust and divide it by Earth's surface area. This can be calculated by multiplying the specific heat of the materials by the mass of material that absorbs solar energy throughout a day and then dividing by surface area. It can be estimated that the specific heat of Earth's materials is a quarter of the specific heat of water, 4,184 J/kgK[4]. We can also approximate that solar energy is absorbed to a depth of 1 m during the course of a day. To determine the mass of the material absorbing solar energy, we can multiply the volume of land that is 1 m deep by the density of Earth materials. The volume of land that is 1 m deep is simply 1 m multiplied by the Earth's surface area. Since we will be dividing this term by Earth's surface area, we can cancel that value. The density of Earth materials is about 1,000 kg/m³. Now we can approximate a value of c for Earth:

$$\begin{aligned} c &\approx \frac{0.25 \cdot \text{specific heat of water} \cdot \text{density of earth materials} \cdot \text{Earth's surface area} \cdot 1 \text{ m}}{\text{Earth's surface area}} \\ &\approx 0.25 \cdot 4,184 \text{ J/kgK} \cdot 1,000 \text{ kg/m}^3 \cdot 1 \text{ m} \\ &\approx 10^6 \text{ J/m}^2\text{K}. \end{aligned}$$

So it would take 10⁶ J of energy to heat one square meter of the Earth's surface by one Kelvin. It turns out that this value of c is a reasonable value to use for Mars as well. A day on Mars lasts 24.6 hours, so it is reasonable to assume that the same volume of land is penetrated by solar radiation during the course of a Martian day. Martian materials are also similar to Earth materials, so it is reasonable to use the same specific heat and density values in a Martian calculation.

We can now use this value for c along with equation (24) to determine the e -folding time for the displacement of the surface temperature from the effective temperature, u on Mars, we get,

$$\tau = \frac{10^6 \text{ J/m}^2\text{K}}{4 (5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4) (217.4 \text{ K})^3} = 4.3 \times 10^5 \text{ s} \approx 5 \text{ days}.$$

This means that it would take 15 days, 3τ , for the displacement to become 95% closer to 0. However, a more sophisticated model would show us that 15 days is not an accurate prediction for the amount of time it would take for the surface temperature of our airless planet to approach its new equilibrium value if the solar constant suddenly rose by 1%. The reason for this is that our c value was calculated by using the time scale of one day and measuring the solar penetration depth during that one day. In reality, if we had chosen another time scale, this depth would be different. A more complex model might look at the planet as having various layers that solar energy can penetrate through time and calculating the rate at which solar energy reaches each consecutive layer. We will not be looking at such a model, so we will not be able to determine accurately how long it would take to reach the new equilibrium temperature. Later, in Chapter 5, we *will* be able to use this value for τ because we will be calculating the diurnal temperature range, which implies a time scale of one day, so having made this calculation was still a useful endeavor.

We will now solve in the in-homogeneous equation. In order to simplify the in-homogeneous equation (22), we will make the following change of variables:

$$\begin{aligned}s &= \frac{2t}{P} \\ ds &= \frac{2}{P} dt \\ \frac{ds}{dt} &= \frac{2}{P}.\end{aligned}$$

Using the chain rule, this tells us that:

$$\begin{aligned}\frac{du}{dt} &= \frac{du}{ds} \cdot \frac{ds}{dt} \\ &= \frac{du}{ds} \cdot \frac{2}{P}.\end{aligned}$$

This allows us to write (22) as a differential equation of u in terms of the new variable, s .

$$(25) \quad \frac{du}{dt} + ku = A\psi\left(\frac{2t}{P}\right)$$

$$\frac{2}{P} \frac{du}{ds} + ku = A\psi(s)$$

$$\frac{du}{ds} + \omega u = B\psi(s)$$

$$(26) \quad \text{where } \omega = \frac{kP}{2} = \frac{P}{2\tau} \text{ and } B = \frac{AP}{2}.$$

We will now find an appropriate integrating factor to multiply (25) by so we can further simplify our equation. Let's try $e^{\omega s}$:

$$\begin{aligned} e^{\omega s} \left(\frac{du}{ds} + \omega u \right) &= \omega e^{\omega s} u(s) + \frac{du}{ds} e^{\omega s} \\ &= \frac{d}{ds} [e^{\omega s} u(s)]. \end{aligned}$$

So $e^{\omega s}$ will work as an integrating factor for (25). We now have:

$$(27) \quad \frac{d}{ds} [e^{\omega s} u(s)] = e^{\omega s} B\psi(s).$$

In order to find the general solution for (25), we can integrate both sides of (27) from 0 to s (first substituting x for s):

$$\begin{aligned} \int_0^s \frac{d}{dx} [e^{\omega x} u(x)] dx &= \int_0^s B e^{\omega x} \psi(x) dx \\ \Rightarrow e^{\omega s} u(s) - u(0) &= B \int_0^s e^{\omega x} \psi(x) dx. \end{aligned}$$

We can now solve this equation for $u(s)$:

$$(28) \quad \begin{aligned} e^{\omega s} u(s) - u(0) &= B \int_0^s e^{\omega x} \psi(x) dx \\ e^{\omega s} u(s) &= B \int_0^s e^{\omega x} \psi(x) dx + u(0) \\ u(s) &= e^{-\omega s} \left[u(0) + B \int_0^s e^{\omega x} \psi(x) dx \right] \\ u(s) &= u(0) e^{-\omega s} + B e^{-\omega s} \int_0^s e^{\omega x} \psi(x) dx. \end{aligned}$$

This general solution to the in-homogeneous equation is in keeping with the concept that the general solution should be equal to:

$$Ce^{-\omega s} + u_p(s)$$

where C is a constant and $u_p(s)$ is a particular solution of the in-homogeneous equation[1].

THEOREM 12. *The following are true:*

(1) *The following equation is periodic and a particular solution to (25).*

$$(29) \quad u(s) = Be^{\omega s} \left[\frac{1}{e^{2\omega} - 1} \int_0^2 e^{\omega x} \psi(x) dx + \int_0^s e^{\omega x} \psi(x) dx \right],$$

(2) *Equation (29) is the unique periodic solution to (25), and*

(3) *$u_g(s) = Ce^{-\omega s} + u_p(s)$, where $u_p(s)$ is equal to equation (29), approaches $u_p(s)$ as $s \rightarrow \infty$.*

PROOF. 1) Let $u(0) = u(2)$, a necessary condition for proving the periodicity of $u(s)$. When we make this substitution in (28), we get:

$$u(0) = e^{-2\omega} u(0) + e^{-2\omega} B \int_0^2 e^{\omega x} \psi(x) dx.$$

Combining the terms with $u(0)$ in them gives us:

$$\begin{aligned} (1 - e^{-2\omega})u(0) &= e^{-2\omega} B \int_0^2 e^{\omega x} \psi(x) dx \\ u(0) &= \frac{e^{-2\omega} B \int_0^2 e^{\omega x} \psi(x) dx}{(1 - e^{-2\omega})} \\ &= \frac{B \int_0^2 e^{\omega x} \psi(x) dx}{e^{2\omega}(1 - e^{-2\omega})} \\ &= \frac{B \int_0^2 e^{\omega x} \psi(x) dx}{e^{2\omega} - 1}. \end{aligned}$$

We can now take this equation for $u(0)$ and substitute it into (28) to get:

$$(30) \quad u(s) = e^{\omega s} \left[\frac{B \int_0^2 e^{\omega x} \psi(x) dx}{e^{2\omega} - 1} + B \int_0^s e^{\omega x} \psi(x) dx \right]$$

$$u(s) = B e^{\omega s} \left[\frac{1}{e^{2\omega} - 1} \int_0^2 e^{\omega x} \psi(x) dx + \int_0^s e^{\omega x} \psi(x) dx \right].$$

We can show that (30) is periodic by showing that $u(s+2) = u(s)$. To do this, we will first evaluate an expression for $u(s+2)$:

$$u(s+2) = B e^{-\omega(s+2)} \left[\frac{1}{e^{2\omega} - 1} \int_0^2 e^{\omega x} \psi(x) dx + \int_0^{s+2} e^{\omega x} \psi(x) dx \right]$$

$$= B e^{-\omega s} e^{-2\omega} \left[(e^{2\omega} - 1)^{-1} \int_0^2 e^{\omega x} \psi(x) dx + \int_0^2 e^{\omega x} \psi(x) dx + \int_2^{s+2} e^{\omega x} \psi(x) dx \right].$$

We will now make the following change of variables in order to simplify the third integral in the above expression:

$$y = x - 2$$

$$\Rightarrow y + 2 = x$$

$$\Rightarrow dy = dx.$$

Because ψ is periodic, we know that $\psi(y+2) = \psi(y)$, so we have:

$$\int_2^{s+2} e^{\omega x} \psi(x) dx = \int_0^s e^{\omega(y+2)} \psi(y+2) dy$$

$$= \int_0^s e^{\omega y} e^{2\omega} \psi(y) dy$$

$$= e^{2\omega} \int_0^s e^{\omega y} \psi(y) dy.$$

Substituting this back into the general solution we get:

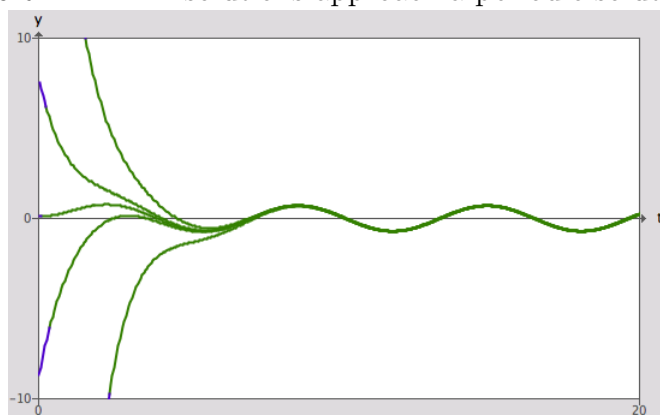
$$\begin{aligned}
 u(s+2) &= Be^{-\omega s} e^{-2\omega} \left[((e^{2\omega} - 1)^{-1} + 1) \int_0^2 e^{\omega x} \psi(x) dx + e^{2\omega} \int_0^2 e^{\omega y} \psi(y) dy \right] \\
 &= Be^{-\omega s} e^{-2\omega} \left[\frac{1 + (e^{2\omega} - 1)}{e^{2\omega} - 1} \int_0^2 e^{\omega x} \psi(x) dx + e^{2\omega} \int_0^s e^{\omega y} \psi(y) dy \right] \\
 &= Be^{-\omega s} e^{-2\omega} \left[e^{2\omega} \left[(e^{2\omega} - 1)^{-1} \int_0^2 e^{\omega x} \psi(x) dx + \int_0^s e^{\omega y} \psi(y) dy \right] \right] \\
 &= Be^{-\omega s} \left[\frac{1}{e^{2\omega} - 1} \int_0^2 e^{\omega x} \psi(x) dx + \int_0^s e^{\omega y} \psi(y) dy \right] \\
 &= u(s).
 \end{aligned}$$

Therefore $u(s)$ as defined in equation (29) is periodic and a particular solution to (25).

2) Because equation (29) was derived using the general solution to (25) and the assumption that $u(0) = u(2)$, a necessary condition for periodicity, it must be the *only* equation that is both periodic and a solution to (25).

3) Because $\omega > 0$, all solutions of (25) will approach the periodic solution, equation (29), when $s \rightarrow \infty$. We know that $u_g(s) = ke^{-\omega s} + u_p(s)$ where $u_g(s)$ is the general solution and $u_p(s)$ is the particular (periodic) solution. As $s \rightarrow \infty$, $ke^{-\omega s} \rightarrow 0$. Therefore, $u_g(s) \rightarrow u_p(s)$. This can be seen in Figure 4.1. \square

FIGURE 4.1. All solutions approach a periodic solution[1].



Now we can use the ψ function as defined in (18) to further simplify the general solution. We will start by solving for $\int_0^s e^{\omega x} \psi(x) dx$ when $0 \leq s \leq 1$ and when

$$1 \leq s \leq 2.$$

Case 1: Suppose $0 \leq s \leq 1$. Then,

$$\begin{aligned} \int_0^s e^{\omega x} \psi(x) dx &= \int_0^s e^{\omega x} dx \\ &= \int_0^{\omega s} e^t \frac{dt}{\omega} \\ &= \frac{1}{\omega} [e^t]_0^{\omega s} \\ &= \frac{1}{\omega} [e^{\omega s} - 1]. \end{aligned}$$

Case 2: Suppose $1 \leq s \leq 2$. Then,

$$\begin{aligned} \int_0^s e^{\omega x} \psi(x) dx &= \int_0^1 e^{\omega x} dx + \int_1^s -e^{\omega x} dx \\ &= \frac{1}{\omega} \int_0^{\omega} e^t dt - \frac{1}{\omega} \int_{\omega}^{\omega s} e^t dt \\ &= \frac{1}{\omega} [e^t]_0^{\omega} - \frac{1}{\omega} [e^t]_{\omega}^{\omega s} \\ &= \frac{1}{\omega} (e^{\omega} - 1) - \frac{1}{\omega} (e^{\omega s} - e^{\omega}) \\ &= \frac{1}{\omega} (2e^{\omega} - e^{\omega s} - 1). \end{aligned}$$

We can use this information to obtain the following by plugging 2 in for s in the above equation:

$$\begin{aligned} \frac{1}{e^{2\omega} - 1} \int_0^2 e^{\omega x} \psi(x) dx &= \frac{1}{e^{2\omega} - 1} \left(\frac{2e^{\omega} - e^{2\omega} - 1}{\omega} \right) \\ &= \frac{2e^{\omega} - 2e^{2\omega} + e^{2\omega} - 1}{\omega(e^{2\omega} - 1)} \\ &= \frac{1}{\omega} \left[\frac{-(2e^{\omega} - 2e^{2\omega})}{-(e^{2\omega} - 1)} + \frac{e^{2\omega} - 1}{e^{2\omega} - 1} \right] \\ &= \frac{1}{\omega} \left[\frac{-2e^{\omega}(1 - e^{\omega})}{(1 - e^{\omega})(1 + e^{\omega})} + 1 \right] \\ &= \frac{1}{\omega} \left[1 - \frac{2e^{\omega}}{1 + e^{\omega}} \right]. \end{aligned}$$

Now that we have all of the integrals worked out, we can define $u(s)$ without using integrals.

Case 1: Suppose $0 \leq s \leq 1$. Then,

$$\begin{aligned}
 (31) \quad u(s) &= Be^{-\omega s} \left[\frac{1}{\omega} \left[1 - \frac{2e^\omega}{1+e^\omega} \right] + \frac{1}{\omega} (e^{\omega s} - 1) \right] \\
 &= \frac{B}{\omega} e^{-\omega s} \left[1 - \frac{2e^\omega}{1+e^\omega} + e^{\omega s} - 1 \right] \\
 &= \frac{B}{\omega} \left[1 - \frac{2e^\omega e^{-\omega s}}{1+e^\omega} \right].
 \end{aligned}$$

Case 2: Suppose $1 \leq s \leq 2$. Then,

$$\begin{aligned}
 (32) \quad u(s) &= Be^{-\omega s} \left[\frac{1}{\omega} \left[1 - \frac{2e^\omega}{1+e^\omega} \right] + \frac{1}{\omega} (2e^\omega - e^{\omega s} - 1) \right] \\
 &= \frac{B}{\omega} e^{-\omega s} \left[1 - \frac{2e^\omega}{1+e^\omega} + 2e^\omega - e^{\omega s} - 1 \right] \\
 &= \frac{B}{\omega} \left[-\frac{2e^\omega e^{-\omega s}}{1+e^\omega} + 2e^\omega e^{-\omega s} - 1 \right] \\
 &= \frac{B}{\omega} \left[-\left(\frac{1}{1+e^\omega} \right) (2e^\omega e^{-\omega s}) - 1 \right] \\
 &= \frac{B}{\omega} \left[\left(1 - \frac{1}{1+e^\omega} \right) (2e^\omega e^{-\omega s}) - 1 \right] \\
 &= \frac{B}{\omega} \left[\frac{1+e^\omega - 1}{1+e^\omega} (2e^\omega e^{-\omega s}) - 1 \right] \\
 &= \frac{B}{\omega} \left[\frac{e^\omega (2e^\omega e^{-\omega s})}{1+e^\omega} - 1 \right] \\
 &= \frac{B}{\omega} \left[\frac{2e^{2\omega} e^{-\omega s}}{1+e^\omega} - 1 \right].
 \end{aligned}$$

So we have:

$$(33) \quad u(s) = \begin{cases} \frac{B}{\omega} \left[1 - \frac{2e^\omega e^{-\omega s}}{1+e^\omega} \right] & , 0 \leq s \leq 1 \\ \frac{B}{\omega} \left[\frac{2e^{2\omega} e^{-\omega s}}{1+e^\omega} - 1 \right] & , 1 \leq s \leq 2. \end{cases}$$

CHAPTER 5

Calculating the Diurnal Temperature Range

5.1. Deriving a General Formula

PROPOSITION 13. *DTR (diurnal temperature range)* $= \frac{2B}{\omega} \left[1 - \frac{2}{1 + e^\omega} \right]$.

PROOF. Using the calculations from Chapter 4, we can obtain a formula for the diurnal temperature range (DTR) of our planet. Because u was defined as $T - T_e$, we can write the DTR as $u_{\max} - u_{\min}$. For $0 \leq s \leq 1$, $u(s)$ is increasing. This is true because as s increases, $e^{-\omega s}$ decreases, so $\frac{2e^\omega e^{-\omega s}}{1 + e^\omega}$ will also decrease resulting in an increase of the expression $1 - \frac{2e^\omega e^{-\omega s}}{1 + e^\omega}$. For $1 \leq s \leq 2$, $u(s)$ is decreasing. This is true because as s increases, $e^{-\omega s}$ decreases, so $\frac{2e^{2\omega} e^{-\omega s}}{1 + e^\omega}$ will also decrease resulting in a decrease of the expression $\frac{2e^{2\omega} e^{-\omega s}}{1 + e^\omega} - 1$. Because $u(s)$ is increasing for $0 \leq s \leq 1$ and $u(s)$ is decreasing for $1 \leq s \leq 2$, $u_{\max} - u_{\min}$ is equivalent to $u(1) - u(0)$. Using (31), we get:

$$\begin{aligned}
 u_{\max} - u_{\min} = u(1) - u(0) &= \frac{B}{\omega} \left(1 - \frac{2}{1 + e^\omega} \right) - \frac{B}{\omega} \left(1 - \frac{2e^\omega}{1 + e^\omega} \right) \\
 &= \frac{B}{\omega} - \frac{B}{\omega} \left(\frac{2}{1 + e^\omega} \right) - \frac{B}{\omega} + \frac{B}{\omega} \left(\frac{2e^\omega}{1 + e^\omega} \right) \\
 &= \frac{2Be^\omega}{\omega(1 + e^\omega)} - \frac{2B}{\omega(1 + e^\omega)} \\
 &= \frac{2B}{\omega} \left[\frac{e^\omega - 1}{1 + e^\omega} \right] \\
 &= \frac{2B}{\omega} \left[\frac{1 - 1 + e^\omega - 1}{1 + e^\omega} \right] \\
 &= \frac{2B}{\omega} \left[1 - \frac{2}{1 + e^\omega} \right].
 \end{aligned}$$

Hence the DTR is given by:

$$(34) \quad \text{DTR} = \frac{2B}{\omega} \left[1 - \frac{2}{1 + e^\omega} \right].$$

□

5.2. Determining a Value for the DTR

We can use the value of c calculated in Chapter 4 because our time scale is irrefutably one day. We will first use some measured scientific information to determine what the DTR on Mars should be. Then we will attempt to calculate it using our model. It has been measured that temperatures on Mars can go as high as about 300 K (80°F) during the day and as low as 160 K (-170°F) at night[4]. This means that the DTR on Mars is about 140 K. Now we will use our model to obtain a similar result.

By (25), we can make the following substitution:

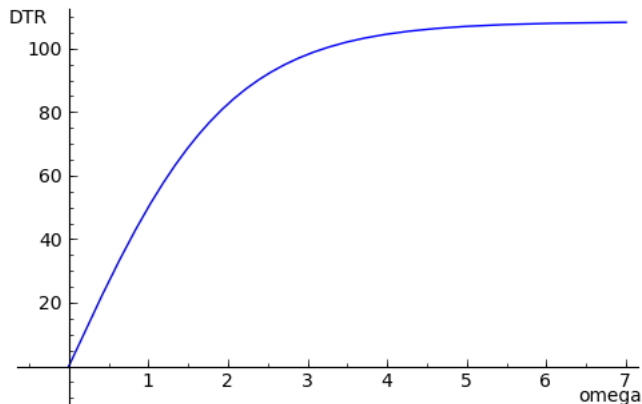
$$(35) \quad \begin{aligned} \frac{B}{\omega} &= \frac{AP/2}{kP/2} \\ &= \frac{A}{k} \\ &= \frac{\sigma T_e^4/c}{4\sigma T_e^3/c} \\ &= \frac{T_e}{4}. \end{aligned}$$

We can use the above substitution to write equation (34) as:

$$(36) \quad \text{DTR} = \frac{T_e}{2} \left[1 - \frac{2}{1 + e^\omega} \right].$$

Figure 5.1 shows us that for a constant T_e , DTR is an increasing function of ω . Recall that $\omega = \frac{kP}{2} = \frac{P}{2\tau}$.

We can also see that the DTR increases to a limit of $T_e/2$. We know this because $\lim_{\omega \rightarrow \infty} \frac{2}{1 + e^\omega} = 0$. This tells us that the DTR can not exceed half of the effective temperature. In the case of Mars, this would mean that the DTR can not exceed 108.45 K (one half of 216.9 K). Although our previous calculation for Mars' DTR was 140 K, we are at least within the same order of magnitude. Our values differ

FIGURE 5.1. $DTR(\omega)$ with $T_e = 216.9$ K

because we have used a simplified assumption that solar intensity does not vary during the course of a day.

It should be noted that the value of 108 K for the Martian DTR is only achieved when ω is large. We will now examine ω more closely to see what it means for ω to be large. By (26) and (24), we can write ω as:

$$(37) \quad \omega = \frac{2\sigma T_e^3 P}{c}.$$

This shows us as c decreases, ω increases and therefore the DTR increases as well. It turns out that this makes physical sense for Mars. The thermal conductivity on Mars is thought to be very small. It has been observed that the Martian surface is very similar to deserts on Earth where only a thin layer of the surface takes part in daily temperature oscillations. Just as the DTR on Earth is larger in deserts than on vegetated land or near water, the DTR on Mars is larger than the DTR on Earth.

Previously we observed algebraically that the DTR is an increasing function of ω . We will now examine why that makes sense physically. By (26), we can think of ω as proportional to P , the planetary rotation period, and inversely proportional to τ , the e -folding time.

Large values of ω imply that the e -folding time is small compared to the length of a day. This means that the day provides ample time for the planet to gain heat and the night provides ample time for the planet to lose heat. Therefore large values of ω should imply large DTRs.

Small values of ω imply that the length of a day is small compared to the e -folding time. This means that there is not enough time during the days or nights for the planet to gain or lose much heat. Therefore small values of ω should imply small DTRs.

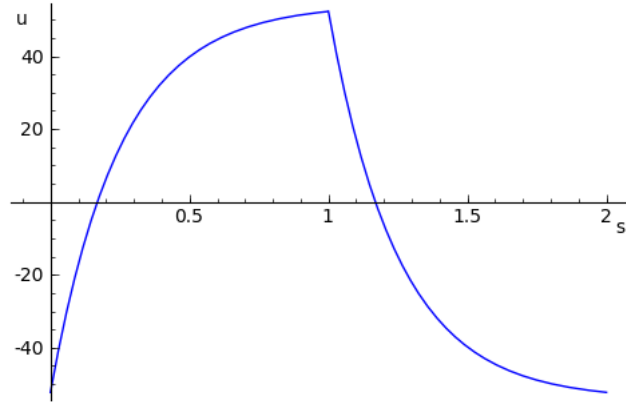
We can use similar physical logic to show that for fixed values of ω , the DTR is an increasing function of T_e (this can be easily observed algebraically). A planet with a larger effective temperature loses heat (via emission of longwave radiation) at a higher rate and so, when the planet is no longer receiving solar radiation at night, it loses heat quickly. Because the effective temperature is high, we know that the planet must be absorbing heat at a higher rate during the day because how else would the planet reach such a high effective temperature? Therefore a higher T_e implies a higher DTR.

At this point, it will be helpful to derive a formula that can express the temperature as a function of time. One way to measure the temperature of a planet is as the difference between its actual temperature and its effective temperature, which can be expressed by the function u . We can use equation (33) along with (35) to write the solution to equation (25) as:

$$(38) \quad u(s) = \begin{cases} \frac{T_e}{4} \left[1 - \frac{2e^\omega e^{-\omega s}}{1 + e^\omega} \right] & , 0 \leq s \leq 1 \\ \frac{T_e}{4} \left[-1 + \frac{2e^\omega e^{-\omega s}}{1 + e^\omega} \right] & , 1 \leq s \leq 2. \end{cases}$$

You will notice in Figure 5.2 that in the interval $0 < s < 1$, u is increasing and concave down and in the interval $1 < s < 2$, u is decreasing and concave up. This can, of course, be explained physically.

At the beginning of the day, when the sun first rises, the planet is still cool from the night. It is therefore emitting energy at a relatively slow rate. It is also absorbing energy at a fast rate because of the solar radiation received during the day, so the planet's temperature rises quickly. Towards the end of the day, the surface is emitting energy at a faster rate because it is hotter, but is still absorbing energy at the same, constant rate as earlier in the day, so the planet's temperature is rising at a slower rate than before.

FIGURE 5.2. $u(s)$ w/ $T_e = 216.9$ 

At dusk, the planet is emitting a lot of heat (built up from the day), but is not receiving any energy from the sun, so its temperature is decreasing at a fast rate. Later during the night, the planet is cooler and is thus not emitting as much energy, so its temperature is still decreasing, but at a slower rate.

5.3. Expressing Changes in the DTR

What happens to the DTR on an airless planet when the effective temperature increases as a result of an increase in the solar constant or for some other reason? There are two ways to answer this question. One way would be to derive an expression for the derivative of the DTR in terms of T_e . To do this, we will first make the following substitution that will help us simplify some of our later calculations:

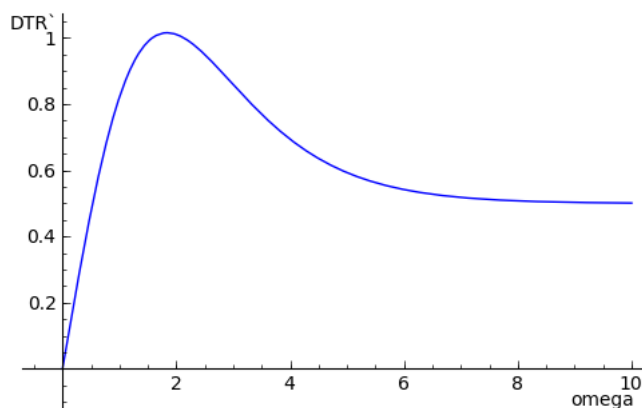
$$\begin{aligned}
 T_e \cdot \frac{d\omega}{dT_e} &= T_e \cdot \left(\frac{6\sigma T_e^2 P}{c} \right) \text{ by (37)} \\
 &= \frac{6\sigma T_e^3 P}{c} \\
 &= 3 \left(\frac{2\sigma T_e^3 P}{c} \right) \\
 &= 3\omega.
 \end{aligned}$$

Using this substitution and the chain rule (by (36), DTR is a function of both T_e and ω), we get:

$$\begin{aligned}
 (39) \quad \frac{d}{dT_e} \text{DTR} &= \left(\frac{\partial \text{DTR}}{\partial T_e} \cdot \frac{dT_e}{dT_e} \right) + \left(\frac{\partial \text{DTR}}{\partial \omega} \cdot \frac{d\omega}{dT_e} \right) \\
 &= \left(\frac{1}{2} - \frac{1}{1+e^\omega} \right) + \left(\frac{T_e e^\omega}{(1+e^\omega)^2} \cdot \frac{3\omega}{T_e} \right) \\
 &= \frac{1}{2} - \frac{1}{1+e^\omega} + \frac{3\omega e^\omega}{(1+e^\omega)^2} \\
 &= \frac{1}{2} \left[1 - \frac{2}{1+e^\omega} + \frac{6\omega e^\omega}{(1+e^\omega)^2} \right].
 \end{aligned}$$

It is clear in Figure 5.3 that the derivative is positive when $\omega > 0$ and thus the

FIGURE 5.3. $\frac{d\text{DTR}}{dT_e}(\omega)$



DTR must be increasing. However, we will now show this mathematically:

PROPOSITION 14. *As T_e increases, DTR will always increase.*

PROOF. The DTR will increase if $\frac{d}{dT_e}\text{DTR} = \frac{1}{2} \left[1 - \frac{2}{1+e^\omega} + \frac{6\omega e^\omega}{(1+e^\omega)^2} \right] > 0$,
so if

$$\begin{aligned} & \frac{1}{2} \left[1 - \frac{2}{1+e^\omega} + \frac{6\omega e^\omega}{(1+e^\omega)^2} \right] > 0, \text{ then} \\ & 1 - \frac{2}{1+e^\omega} + \frac{6\omega e^\omega}{(1+e^\omega)^2} > 0 \\ & \Leftrightarrow \frac{-2(1+e^\omega) + 6\omega e^\omega}{(1+e^\omega)^2} > -1 \\ & \Leftrightarrow -2 - 2e^\omega + 6\omega e^\omega > -(1+e^\omega)^2 \\ & \Leftrightarrow -2 + (6\omega - 2)e^\omega > -(1 + 2e^\omega + e^{2\omega}) \\ & \Leftrightarrow -2 + (6\omega - 2)e^\omega > -1 - 2e^\omega - e^{2\omega} \\ & \Leftrightarrow -1 + 6\omega e^\omega + e^{2\omega} > 0. \end{aligned}$$

Because $\forall \omega > 0$, $e^{2\omega} > 1$ and thus $-1 + e^{2\omega} > 0$, we can say that:

$$-1 + 6\omega e^\omega + e^{2\omega} > 6\omega e^\omega > 0.$$

$6\omega e^\omega$ is always greater than 0, so because we initially defined ω to be a positive value, this shows that as T_e increases, DTR will always increase. Another, quicker way to show this would be to think about the fact that DTR increases as ω increases, equation (34), and ω increases as T_e increases, equation (37), so as T_e increases, DTR will increase. \square

CHAPTER 6

Effects of Atmospheric Presence

As discussed in Chapter 2, in reality the Earth's average surface temperature is warmer than its effective temperature by about 35 K. It has also been mentioned that this difference is caused by the presence of an atmosphere, which we have previously assumed to be non-existent. Because of the longwave and shortwave radiation that is emitted and absorbed by the atmosphere (also discussed in Chapter 2), the atmosphere keeps the surface temperature on Earth warmer than it would be in the absence of an atmosphere. We will now look closer at these different kinds of radiation. To do that we will assume that every point on Earth has temperature equal to T_0 where T_0 is a function of time, so $T_0 = T_0(t)$. Similarly we will assume that every point in the atmosphere has temperature equal to T_1 where T_1 is a function of time, so $T_1 = T_1(t)$. There are three types of power influencing the differences in temperature through time:

- shortwave radiative power
- longwave radiative power
- nonradiative power

6.1. Shortwave Radiative Power

Recall from (2) that the Flux In from the Sun is $a\Omega/4$. With the addition of the atmosphere, this value will stay the same, but it will now represent the total flux from the Sun received by the Earth/atmosphere system. Part of that flux will be absorbed by the atmosphere and part of it will be absorbed by the Earth. It should also be noted that a flux of $\Omega/4$ reaches the edge of the atmosphere of which a fraction, $\alpha = 1 - a$, is reflected back into space (recall that α is the Earth's albedo defined in Definition 3). Of the flux that penetrates the atmosphere, a fraction, q , is absorbed by the surface (after first undergoing a series of reflections back and forth between the atmosphere and the surface). That means that a fraction, $1 - q$, of the flux that penetrates the atmosphere is absorbed by the atmosphere (after also

undergoing a similar series of reflections). By (2), we now have:

$$(40) \quad \text{shortwave radiative power absorbed by surface} \quad = Sq\sigma T_e^4,$$

$$(41) \quad \text{shortwave radiative power absorbed by atmosphere} \quad = S(1 - q)\sigma T_e^4.$$

In this case, S is the Earth's surface area. We multiply by S because flux represents the power/unit area, so to get the total power, we need to multiply by the Earth's surface area.

6.2. Longwave Radiative Power

Our previous assumption was that the Earth's surface behaves like a blackbody, an object that absorbs 100% of the radiation that it receives. This is approximately the case in the longwave range. At typical surface and atmospheric temperatures, essentially all of the emitted radiation is also emitted in the longwave range[4]. Based on this approximation, we can say that the surface flux emitted is σT_0^4 . Since the atmosphere does not behave like a blackbody, we cannot assume that its flux is σT_1^4 . The atmosphere does not behave like a blackbody because it absorbs a fraction A , where $0 \leq A \leq 1$, of the flux σT_0^4 received from the surface.

DEFINITION 15. A is the atmosphere's absorptivity.

The atmosphere also emits a fraction ϵ , where $0 \leq \epsilon \leq 1$, of the flux it would emit if it behaved like a blackbody (σT_1^4).

DEFINITION 16. ϵ is the atmosphere's emissivity.

For the purposes of our model we can assume that $A = \epsilon$. We can make this assumption because of Kirchoff's Law, which states that 'the efficiency of a body at absorbing radiation at a given wavelength equals its efficiency at emitting radiation at that wavelength'[4]. It is important to note that the wavelengths of the absorbed and emitted radiation must be equal for Kirchoff's Law to hold.

To come up with expressions for longwave radiative power absorbed and emitted by the surface and the atmosphere, we need to think of the atmosphere as two spheres, one inside the other. The smaller sphere interacts directly with the Earth's surface and the larger sphere interacts directly with outer space. We can consider the surface area of both of these spheres to be equivalent because, compared to the size of the Earth, the thickness of the atmosphere is negligible. The atmosphere

is as thin compared to the Earth as the skin of an apple is to an apple. We will consider the surface area of both spheres to be equal to S , the surface area of the Earth. With this information, we can now derive the following expressions:

$$(42) \quad \text{longwave radiative power absorbed by surface} \quad = S\epsilon\sigma T_1^4,$$

$$(43) \quad \text{longwave radiative power emitted by surface} \quad = S\sigma T_0^4,$$

$$(44) \quad \text{longwave radiative power absorbed by atmosphere} \quad = S\epsilon\sigma T_0^4,$$

$$(45) \quad \text{longwave radiative power emitted by atmosphere} \quad = 2S\epsilon\sigma T_1^4.$$

In equations (42) and (45), the longwave radiation emitted from the atmosphere is being referenced. The reason why (45) is twice as large as (42) is that radiation is emitted in two directions from the atmosphere. Half of it is emitted out to space and half of it is emitted to and then consequently absorbed by the surface. Equations (43) and (44) are also related. While the surface emits $S\sigma T_0^4$ units of power, only a fraction ϵ of that power is absorbed by the atmosphere while $(1 - \epsilon)S\sigma T_0^4$ units of power are emitted from the Earth's surface all the way out into space.

6.3. Nonradiative Power

There are two ways that energy is transferred between the surface and the atmosphere that do not involve radiation. They are: conduction in which heat is transferred from the Earth's surface to the bottom layer of the atmosphere and then to subsequent, higher levels, and via evaporating water in which water vapor evaporated from Earth's surface releases energy into the atmosphere when it condenses at high altitudes. For the purposes of our model, we will simplify these processes into one general process that we will call 'mechanical heat transfer.' The expression that we will use to describe mechanical heat transfer is derived using Newton's law of cooling, which states that the rate at which energy is transferred mechanically between two bodies is proportional to the difference between the temperatures of those bodies, so if $\gamma > 0$ is a constant, this gives us:

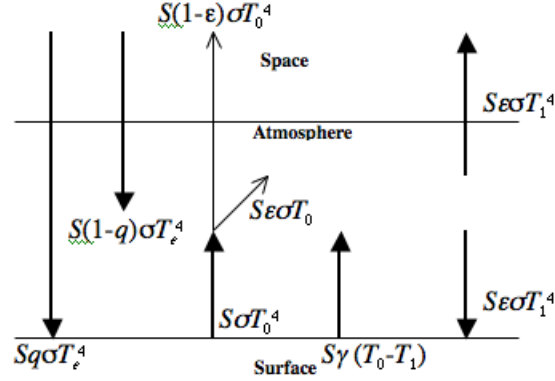
$$(46) \quad \text{power leaving surface by mechanical means} \quad = S\gamma(T_0 - T_1),$$

$$(47) \quad \text{power entering the atmosphere by mechanical means} \quad = S\gamma(T_0 - T_1).$$

Later we will show that, for the purposes of our model, we can assume that $\gamma = 0$.

Figure 6.1 summarizes all of the power entering, leaving, and being exchanged within the surface-atmosphere system.

FIGURE 6.1. Diagram of energy flows for the surface-atmosphere system



6.4. Equilibrium Conditions

Now that we have a sense of how energy is transmitted between the surface, the atmosphere, and space, we can begin to discuss the equilibrium conditions of our planetary system. We will denote the equilibrium surface and atmospheric temperatures as \hat{T}_0 and \hat{T}_1 respectively. As we've done before, we can derive expressions for \hat{T}_0 and \hat{T}_1 by setting the sum of all of the power absorbed by the surface (or atmosphere, as the case may be) equal to the sum of all of the powers emitted by the surface (or atmosphere). By adding incoming and outgoing terms from equations (40) - (47) this gives us:

$$(48) \quad q\sigma T_e^4 + \epsilon\sigma\hat{T}_1^4 = \sigma\hat{T}_0^4 + \gamma(\hat{T}_0 - \hat{T}_1)$$

$$(49) \quad (1 - q)\sigma T_e^4 + \epsilon\sigma\hat{T}_0^4 + \gamma(\hat{T}_0 - \hat{T}_1) = 2\epsilon\sigma\hat{T}_1^4,$$

where equation (48) represents the balance of powers for the surface and equation (49) represents the balance of powers for the atmosphere. We can combine equations (48) and (49), which will allow us to derive expressions for our coefficients as well

as for our equilibrium temperature values:

$$(50) \quad \text{Total (Surface + Atmosphere) Power In} = \text{Total Power Out}$$

$$q\sigma T_e^4 + \epsilon\sigma\hat{T}_1^4 + (1-q)\sigma T_3^4 + \epsilon\sigma\hat{T}_0^4 + \gamma(\hat{T}_0 - \hat{T}_1) = \sigma\hat{T}_0^4 + \gamma(\hat{T}_0 - \hat{T}_1) + 2\epsilon\sigma\hat{T}_1^4$$

$$\sigma T_e^4 = (1-\epsilon)\sigma\hat{T}_0^4 + \epsilon\sigma\hat{T}_1^4.$$

We can now solve for ϵ :

$$(51) \quad \sigma T_e^4 = (1-\epsilon)\sigma\hat{T}_0^4 + \epsilon\sigma\hat{T}_1^4$$

$$\sigma T_e^4 = \sigma\hat{T}_0^4 - \epsilon\sigma(\hat{T}_0^4 + \hat{T}_1^4)$$

$$\sigma(T_e^4 - \hat{T}_0^4) = -\epsilon\sigma(\hat{T}_0^4 + \hat{T}_1^4)$$

$$\frac{T_e^4 - \hat{T}_0^4}{\hat{T}_0^4 + \hat{T}_1^4} = -\epsilon$$

$$\epsilon = \frac{\hat{T}_0^4 - T_e^4}{\hat{T}_0^4 + \hat{T}_1^4}.$$

Using equation (48), we can also solve for q :

$$(52) \quad q\sigma T_e^4 + \epsilon\sigma\hat{T}_1^4 = \sigma\hat{T}_0^4 + \gamma(\hat{T}_0 - \hat{T}_1)$$

$$q\sigma T_e^4 = \sigma\hat{T}_0^4 - \epsilon\sigma\hat{T}_1^4 + \gamma(\hat{T}_0 - \hat{T}_1)$$

$$q = \frac{\hat{T}_0^4 - \epsilon\hat{T}_1^4 + \gamma\sigma^{-1}(\hat{T}_0 - \hat{T}_1)}{T_e^4},$$

as well as for γ :

$$(53) \quad q\sigma T_e^4 + \epsilon\sigma\hat{T}_1^4 = \sigma\hat{T}_0^4 + \gamma(\hat{T}_0 - \hat{T}_1)$$

$$q\sigma T_e^4 + \epsilon\sigma\hat{T}_1^4 - \sigma\hat{T}_0^4 = \gamma(\hat{T}_0 - \hat{T}_1)$$

$$\gamma = \frac{\sigma(qT_e^4 + \epsilon\hat{T}_1^4 - \hat{T}_0^4)}{\hat{T}_0 - \hat{T}_1}.$$

Using the following data[4], we can calculate numerical values for ϵ and q . In our model we will assume that $\gamma = 0$, so we will be neglecting mechanical heat flow:

- $T_e = 255$ K,
- $\hat{T}_0 = 290$ K, and
- $\hat{T}_1 = 250$ K.

Using these temperature values that are measured values for present day Earth, we get the following numerical values for ϵ and q (with $\gamma = 0$):

$$(54) \quad \epsilon = 0.8983 \quad \text{and} \quad q = 0.8428.$$

We can now take the information we have and derive a function for the equilibrium surface temperature (\hat{T}_0) in terms of the effective temperature (T_e), ϵ , and q .

PROPOSITION 17.

$$(55) \quad \hat{T}_0 = T_e G(\epsilon, q) \text{ where } G(\epsilon, q) = \left(\frac{1+q}{2-\epsilon} \right)^{1/4}.$$

PROOF. We will start with equations (48) and (49) with $\gamma = 0$:

$$\text{Rearranging equation (48)} \Rightarrow q\sigma T_e^4 = \sigma \hat{T}_0^4 - \epsilon\sigma \hat{T}_1^4$$

$$(1-q)\sigma T_e^4 + \epsilon\sigma \hat{T}_0^4 = 2\epsilon\sigma \hat{T}_1^4$$

$$\text{Multiplying equation (48) by two} \Rightarrow 2q\sigma T_e^4 = 2\sigma \hat{T}_0^4 - 2\epsilon\sigma \hat{T}_1^4$$

$$(1-q)\sigma T_e^4 + \epsilon\sigma \hat{T}_0^4 = 2\epsilon\sigma \hat{T}_1^4$$

$$\text{Adding equations (48) and (49)} \Rightarrow (1+q)\sigma T_e^4 + \epsilon\sigma \hat{T}_0^4 = 2\sigma \hat{T}_0^4$$

$$\Rightarrow (1+q)\sigma T_e^4 = 2\sigma \hat{T}_0^4 - \epsilon\sigma \hat{T}_0^4$$

$$\Rightarrow (1+q)T_e^4 = \hat{T}_0^4(2-\epsilon)$$

$$\Rightarrow \left(\frac{1+q}{2-\epsilon} \right) T_e^4 = \hat{T}_0^4$$

$$\Rightarrow \hat{T}_0 = T_e G(\epsilon, q),$$

$$\text{where } G(\epsilon, q) = \left(\frac{1+q}{2-\epsilon} \right)^{1/4}.$$

□

As you may recall, without an atmosphere $T_e = \hat{T}_0$. This would imply that without an atmosphere, $G(\epsilon, q) = 1$. The function G represents the factor by which the equilibrium surface temperature increases or decreases in relation to the effective temperature (G can be more or less than 1). Because of the relationship between G and a planet's atmosphere, we call G the greenhouse function.

6.5. Impacts of Changing The Atmospheric Composition

One question that is commonly discussed in climate modeling is what is the effect of CO₂ and other greenhouse gasses on surface temperatures. It turns out that, to a certain extent, we can use our model to answer this question. Although greenhouse gasses have, at the very least, an indirect effect on essentially all of the variables and parameters in our model, the parameter that is most effected is ϵ , the absorptivity/emissivity of the atmosphere. The higher the concentration of CO₂ in the atmosphere, the bigger ϵ will be because the atmosphere will be more efficient at absorbing and emitting longwave radiation. That is why greenhouse gasses have a general warming effect. They keep the energy that the Earth would otherwise radiate out into space trapped in the atmosphere as well as re-radiating that energy back to the Earth's surface.

Before the industrial revolution, there was a concentration of CO₂ in our atmosphere of about 280 ppm(v) (parts per million by volume) meaning that for every million atmospheric volume units, 280 of those units were CO₂[4]. We would like to determine what would happen to the global surface temperature if this pre-industrial concentration were doubled to 560 ppm(v). In 2009 it was measured that the concentration was measured at about 385 ppm(v)[3].

PROPOSITION 18. *The Earth's equilibrium surface temperature, \hat{T}_0 , will increase by about 0.9 K from before the industrial revolution to the time when pre-industrial CO₂ levels have doubled.*

PROOF. We can calculate the temperature increase by using the above information to quantitatively determine a pre-industrial value of ϵ and a value of ϵ for the future when pre-industrial CO₂ concentrations have doubled. To do this, we will consider the two greenhouse gasses that exist in the highest concentrations in the atmosphere: CO₂ and water vapor. It is a fair assumption that the ability of a composite of gasses to absorb and emit radiation is equal to the sums of the abilities of each gas to individually absorb and emit radiation. Therefore we will assume that

$$\epsilon = \epsilon_w + \epsilon_c,$$

where ϵ_w is the absorptivity/emissivity of atmospheric water vapor and ϵ_c is the absorptivity/emissivity of atmospheric carbon dioxide. We will also assume that the concentration of water vapor is directly proportional to its ability to absorb

and emit longwave radiation, so $\epsilon_w = \beta(\text{concentration of water vapor})$ where β is a constant. Carbon dioxide behaves in a similar way, with the important exception that CO_2 molecules are one quarter as effective at absorbing and emitting longwave radiation as water vapor molecules, so $\epsilon_c = \beta(\text{concentration of water vapor})/4$. It has been shown that the concentration of water vapor in the atmosphere is currently about 14.6 times the concentration of carbon dioxide[4]. This would mean that there is currently a water vapor concentration of $14.6 \times 385 \text{ ppm(v)} = 5621 \text{ ppm(v)}$.

The next assumption that we make is not entirely based in reality, but we will make this assumption anyway because it is both mathematically convenient and it allows us to isolate the effect of CO_2 emissions on global temperature. We will now assume that the concentration of water vapor in the atmosphere is constant through time. In fact it is not only variable through time, but it also varies from region to region across the globe.

We can now use all of this information to determine a value for ‘future ϵ ,’ i.e. the absorptivity/emissivity of the atmosphere after pre-industrial CO_2 concentrations have doubled and ‘past ϵ ,’ i.e. the absorptivity/emissivity of the atmosphere before the industrial revolution:

$$(56) \quad \begin{aligned} \frac{\text{future } \epsilon}{\text{current } \epsilon} &= \frac{\beta[(\text{current water vapor}) + 0.25(\text{future } \text{CO}_2)]}{\beta[(\text{current water vapor}) + 0.25(\text{current } \text{CO}_2)]} \\ &= \frac{5621 \text{ ppm(v)} + 0.25 \cdot 560 \text{ ppm(v)}}{5621 \text{ ppm(v)} + 0.25 \cdot 385 \text{ ppm(v)}} \\ &\approx 1.0077 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \text{future } \epsilon &= 1.0077(\text{current } \epsilon) \\ &= 1.0077(0.8983) = 0.9052, \text{ and} \end{aligned}$$

$$(57) \quad \begin{aligned} \frac{\text{past } \epsilon}{\text{current } \epsilon} &= \frac{\beta[(\text{current water vapor}) + 0.25(\text{past } \text{CO}_2)]}{\beta[(\text{current water vapor}) + 0.25(\text{current } \text{CO}_2)]} \\ &= \frac{5621 \text{ ppm(v)} + 0.25 \cdot 280 \text{ ppm(v)}}{5621 \text{ ppm(v)} + 0.25 \cdot 385 \text{ ppm(v)}} \\ &\approx 0.9954 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \text{past } \epsilon &= 0.9954(\text{current } \epsilon) \\ &= 0.9954(0.8953) = 0.8912. \end{aligned}$$

Using these values for past and future ϵ , we can use equation (55) to determine the equilibrium surface temperature at each of those times assuming that q and T_e remain constant with q at its present day value:

$$\begin{aligned}
\hat{T}_0^{past} &= T_e G(\text{past } \epsilon, q) \\
&= T_e \left(\frac{1+q}{2-\text{past } \epsilon} \right)^{1/4} \\
&= 255 \text{ K} \left(\frac{1+0.8428}{2-0.8912} \right)^{1/4} \\
&= 289.53 \text{ K, and} \\
\hat{T}_0^{future} &= T_e G(\text{future } \epsilon, q) \\
&= T_e \left(\frac{1+q}{2-\text{future } \epsilon} \right)^{1/4} \\
&= 255 \text{ K} \left(\frac{1+0.8428}{2-0.9052} \right)^{1/4} \\
&= 290.45 \text{ K.}
\end{aligned}$$

This gives us a temperature increase of about 0.9 K. □

We will now work towards proving the following theorem that was mentioned earlier in this section:

THEOREM 19. *For our model, we can assume that $\gamma = 0$ because γ has little effect on the equilibrium surface temperature.*

PROOF. We will assume that $q = 1$ and $\gamma > 0$. By solving equation (50) for \hat{T}_1 and plugging that expression into equation (49) with $q = 1$, we will start by showing that \hat{T}_0 is a zero of the following function:

$$(58) \quad f(x) = 2\sigma T_e^4 - (2-\epsilon)\sigma x^4 - \gamma \left[x - \left(\frac{T_e^4 - (1-\epsilon)x^4}{\epsilon} \right)^{1/4} \right].$$

Solving equation (50) for \hat{T}_1 gives us:

$$\begin{aligned}\sigma T_e^4 &= (1 - \epsilon)\sigma \hat{T}_0^4 + \epsilon\sigma \hat{T}_1^4 \\ \sigma T_e^4 - (1 - \epsilon)\sigma \hat{T}_0^4 &= \epsilon\sigma \hat{T}_1^4 \\ \hat{T}_1^4 &= \frac{T_e^4 - (1 - \epsilon)\hat{T}_0^4}{\epsilon} \\ \hat{T}_1 &= \left(\frac{T_e^4 - (1 - \epsilon)\hat{T}_0^4}{\epsilon} \right)^{1/4}.\end{aligned}$$

Plugging this into equation (49) gives us:

$$\begin{aligned}\epsilon\sigma \hat{T}_0^4 + \gamma \left[\hat{T}_0 - \left(\frac{T_e^4 - (1 - \epsilon)\hat{T}_0^4}{\epsilon} \right)^{1/4} \right] &= 2\epsilon\sigma \left[\left(\frac{T_e^4 - (1 - \epsilon)\hat{T}_0^4}{\epsilon} \right)^{1/4} \right]^4 \\ \epsilon\sigma \hat{T}_0^4 + \gamma \left[\hat{T}_0 - \left(\frac{T_e^4 - (1 - \epsilon)\hat{T}_0^4}{\epsilon} \right)^{1/4} \right] &= 2\sigma(T_e^4 - (1 - \epsilon)\hat{T}_0^4) \\ \epsilon\sigma \hat{T}_0^4 + \gamma \left[\hat{T}_0 - \left(\frac{T_e^4 - (1 - \epsilon)\hat{T}_0^4}{\epsilon} \right)^{1/4} \right] &= 2\sigma T_e^4 - 2\sigma(1 - \epsilon)\hat{T}_0^4 \\ 2\sigma T_e^4 - (2 - \epsilon)\sigma \hat{T}_0^4 - \gamma \left[\hat{T}_0 - \left(\frac{T_e^4 - (1 - \epsilon)\hat{T}_0^4}{\epsilon} \right)^{1/4} \right] &= 0.\end{aligned}$$

So \hat{T}_0 is a zero of equation (58). Below we will show that $f(x)$ is a decreasing function, which will prove that not only is \hat{T}_0 a root of f , but it is, in fact the *only* root of f .

We can tell that f is decreasing by examining each of the terms that depend on x within the function:

- $-(2 - \epsilon)\sigma x^4$
 - * As x increases, this expression always decreases.
- $-\gamma \left[x - \left(\frac{T_e^4 - (1 - \epsilon)x^4}{\epsilon} \right)^{1/4} \right]$
 - * The term $-\gamma x$ will decrease as x increases.
 - * The term $-(1 - \epsilon)x^4$ will decrease as x increases.
 - * The term $-(T_e^4 - (1 - \epsilon)x^4)$ will increase as x increases.

* This means that the term $+\gamma \left(\frac{T_e^4 - (1 - \epsilon)x^4}{\epsilon} \right)^{1/4}$ will decrease as x increases.

Since every term that depends on x is decreasing, the function as a whole must be decreasing.

We can now use equation (53) to numerically determine the unique value of γ . We use $q = 1$ and the present day values for ϵ , $\epsilon = 0.8983$, T_e , $T_e = 255$ K, \hat{T}_1 , $\hat{T}_1 = 250$ K, and \hat{T}_0 , $\hat{T}_0 = 290$ K:

$$\gamma = \frac{\sigma(qT_e^4 + \epsilon\hat{T}_1^4 - \hat{T}_0)}{\hat{T}_0 - \hat{T}_1} = 0.9418 \text{ W/m}^2\text{K}.$$

Using this value for γ , we can find the root of f using the past ϵ , (57), and the future ϵ , (56), which we can use to calculate the equilibrium surface temperatures for each time period. Because of the complexity of f , I used the NSolve command in Mathematica to determine that, for the past ϵ , $\hat{T}_0 = 289.554$ and for the future ϵ , $\hat{T}_0 = 290.437$. This shows a difference of 0.883 K ≈ 0.9 K.

Therefore, it makes little difference to the overall outcome of the model to let $\gamma = 0$. □

CHAPTER 7

Changes in the DTR with Atmospheric Presence

Recall from our discussion about a planet without an atmosphere that we used the power balance equation to derive a differential equation for the equilibrium temperature with respect to time. We can use our surface and atmospheric power balance equations, (48)-(49), to do the same thing for a planet that does have an atmosphere.

We will continue to assume that $\gamma = 0$. First we will work with the surface balance equation. Recall that C_0 is the surface's heat capacity, S is the planet's surface area, and $c_0 = C_0/S$ is the surface's heat capacity per square meter. From Chapter 3 we know that $T_e^4(t) = T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right]$. Then:

$$\begin{aligned}
 (59) \quad C_0 \frac{dT_0}{dt} &= \text{Power In} - \text{Power Out} \\
 C_0 \frac{dT_0}{dt} &= S(q\sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] + \epsilon\sigma T_1^4) - S(\sigma T_0^4) \\
 c_0 S \frac{dT_0}{dt} &= S q \sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] + S \epsilon \sigma T_1^4 - S \sigma T_0^4 \\
 c_0 \frac{dT_0}{dt} &= q \sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] + \epsilon \sigma T_1^4 - \sigma T_0^4.
 \end{aligned}$$

It should be noted that ψ is as it was defined in Chapter 3 and refers to the variance in solar flux throughout the course of a day.

We will now use similar logic to derive a differential equation for the equilibrium atmospheric temperature. Now C_1 is the atmosphere's heat capacity and $c_1 = C_1/S$

is the atmosphere's heat capacity per square meter:

$$(60) \quad \begin{aligned} C_1 \frac{dT_1}{dt} &= S(1-q)\sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] + S\epsilon\sigma T_0^4 - 2S\epsilon\sigma T_1^4 \\ c_1 \frac{dT_1}{dt} &= (1-q)\sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] + \epsilon\sigma T_0^4 - 2\epsilon\sigma T_1^4 \\ c_1 \frac{dT_1}{dt} &= (1-q)\sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] + \epsilon\sigma T_0^4 - 2\epsilon\sigma T_1^4. \end{aligned}$$

We will now assume that T_1 is constant: $T_1(t) = \hat{T}_1, \forall t$. We can make this assumption because the atmosphere experiences relatively small diurnal temperature fluctuations compared to the fluctuations on the surface. Using this assumption, we will linearize our differential equation for the surface temperature as we have done before in terms of u where, this time, $u = T_0 - \hat{T}_0$. We will also change to the independent variable $s = 2t/P$, so we'll have $ds = \frac{2}{P}dt$. Simplifying equation (59) gives us:

$$\begin{aligned} c_0 \frac{dT_0}{dt} &= q\sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] + \epsilon\sigma \hat{T}_1^4 - \sigma T_0^4 \\ c_0 \frac{d(\hat{T}_0 + u)}{dt} &= q\sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] + \epsilon\sigma \hat{T}_1^4 - \sigma(\hat{T}_0 + u)^4 \\ c_0 \frac{du}{dt} &= q\sigma T_e^4 \left[1 + \psi \left(\frac{2t}{P} \right) \right] + \epsilon\sigma \hat{T}_1^4 - \sigma(\hat{T}_0^4 + 4\hat{T}_0^3 u) \\ c_0 \frac{du}{dt} &= q\sigma T_e^4 + q\sigma T_e^4 \psi \left(\frac{2t}{P} \right) + \epsilon\sigma \hat{T}_1^4 - \sigma \hat{T}_0^4 - 4\sigma \hat{T}_0^3 u \\ c_0 \frac{du}{ds} \left(\frac{2}{P} \right) &= q\sigma T_e^4 + q\sigma T_e^4 \psi(s) + \epsilon\sigma \hat{T}_1^4 - \sigma \hat{T}_0^4 - 4\sigma \hat{T}_0^3 u \\ \frac{du}{ds} &= \frac{P\sigma}{2c_0} (qT_e^4 + qT_e^4 \psi(s) + \epsilon \hat{T}_1^4 - \hat{T}_0^4 - 4\hat{T}_0^3 u) \\ \frac{du}{ds} + \frac{2P\sigma \hat{T}_0^3}{c_0} u &= \frac{P\sigma}{2c_0} (qT_e^4 + \epsilon \hat{T}_1^4 - \hat{T}_0^4) + \frac{Pq\sigma T_e^4}{2c_0} \psi(s). \end{aligned}$$

Since $q = \frac{\hat{T}_0^4 - \epsilon \hat{T}_1^4}{T_e^4}$, we have:

$$(61) \quad \frac{du}{ds} + \frac{2P\sigma\hat{T}_0^3}{c_0}u = \frac{P\sigma}{2c_0}(\hat{T}_0^4 - \epsilon\hat{T}_1^4 + \epsilon\hat{T}_1^4 - \hat{T}_0^4) + \frac{Pq\sigma T_e^4}{2c_0}\psi(s).$$

$$\text{Hence, } \frac{du}{ds} + \omega u = B\psi(s),$$

$$(62) \quad \text{where } \omega = \frac{2P\sigma\hat{T}_0^3}{c_0} \text{ and } B = \frac{Pq\sigma T_e^4}{2c_0}.$$

Note that, although the expressions for ω and B are slightly different, this is the same as equation (25). Recall that given the ψ described in equation (18), the diurnal temperature range (DTR) can be described as follows:

$$(63) \quad \text{DTR} = \frac{2B}{\omega}f(\omega), \text{ where } f(\omega) = 1 - \frac{2}{1 + e^\omega}.$$

We can now use this equation to determine if and how the presence of an atmosphere affects a planet's DTR. Although there are many ways that an atmosphere may affect a planet's DTR, we are going to focus specifically on the effect that ϵ , the atmosphere's absorptivity and emissivity, has on DTR. To do this, we will compare two planets with identical rotation period (P), effective temperature (T_e), surface area (S) and heat capacity (C), but with one planet possessing an atmosphere and the other being atmosphere-free.

For the planet without an atmosphere, $\epsilon = 0$ and $q = 1$. We know $q = 1$ because, without an atmosphere, 100% of the sun's radiation that reaches the Earth, is absorbed by the surface and $\epsilon = 0$ because without an atmosphere, there can be no amount of radiation absorbed or emitted by the atmosphere.

Using equation (62), we can define ω_0 to be the ω value for our atmosphere-free planet, so $\omega_0 = \frac{2P\sigma T_e^3}{c_0}$. Recall from equation (3) that $\hat{T}_0 = T_e$ when there is no atmospheric presence.

We can now determine an expression for ω on the planet with an atmosphere using equation (62) and our greenhouse gas function from Chapter 8:

$$\begin{aligned}\omega &= \frac{2P\sigma\hat{T}_0^3}{c_0} \\ &= \frac{2P\sigma T_e^3 [G(\epsilon, q)]^3}{c_0} \\ &= [G(\epsilon, q)]^3 \omega_0.\end{aligned}$$

We can now write an expression for the ratio of the DTR of a planet with an atmosphere to the DTR of a planet without an atmosphere:

$$(64) \quad \frac{\frac{2B}{\omega} f(\omega)}{\frac{2B}{\omega_0} f(\omega_0)} = [G(\epsilon, q)]^{-3} F(\omega_0),$$

where

$$(65) \quad F(x) = f([G(\epsilon, q)]^3 x) / f(x).$$

We noted in Chapter 6 that at the moment the Earth has ϵ and q values such that $\epsilon + q > 1$. Given that $\epsilon + q > 1$, we can discuss some interesting properties of the function $F(x)$ that will help us to visualize how the ratio of DTRs changes as ω_0 increases.

(1) We can see that $F(x) \rightarrow [G(\epsilon, q)]^3$ as $x \rightarrow 0^+$ because:

(a) $\frac{f([G(\epsilon, q)]^3 x)}{f(x)} \rightarrow \frac{0}{0}$ as $x \rightarrow 0^+$, and

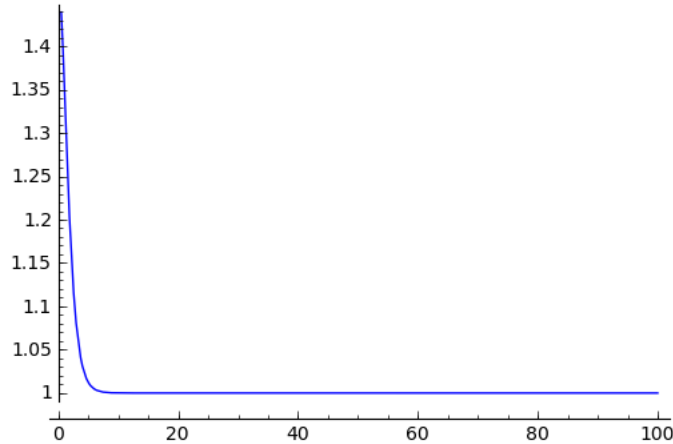
(b) We can use L'Hopitals Rule[6] to get $\frac{f'([G(\epsilon, q)]^3 x)}{f'(x)} \rightarrow \frac{[G(\epsilon, q)]^3 \cdot \frac{1}{2}}{\frac{1}{2}} = [G(\epsilon, q)]^3$ as $x \rightarrow 0^+$.

(2) Also, F is a decreasing function.

(a) This can be seen in Figure 7.1, a graph of $F(x)$. (Recall that $x \not\prec 0$).

(3) Also note that $F(x) \rightarrow 1$ as $x \rightarrow \infty$ by the following logic:

(a) $\frac{1 - \frac{2}{1 + e^{[G(\epsilon, q)]^3 x}}}{1 - \frac{2}{1 + e^x}} \rightarrow \frac{1}{1} = 1$ as $x \rightarrow \infty$.

FIGURE 7.1. $F(x)$ as defined in equation (65) for $0 \leq x \leq 100$ 

This information tells us a few important things about the DTR on planets with and without an atmosphere. First of all, since $1 < F(x) < [G(\epsilon, q)]^3$, the ratio of the atmosphere-free DTR to the DTR on a planet with an atmosphere is between 0 and 1. This means that the atmosphere-free planet will always have a higher DTR than the planet with an atmosphere. Also, since the cube of the greenhouse function is a constant in the denominator of equation (64), $[G(\epsilon, q)]^3$ represents the maximum amount of compression that the DTR could undergo given the addition of an atmosphere. For instance, for the present values for the Earth, $\epsilon = 0.8983$ and $q = 0.8428$, we can calculate the factor needed to assess what the DTR on an Earth-like planet would be if it had no atmosphere:

$$[G(\epsilon, q)]^3 = [G(0.8983, 0.8428)]^3 = \left[\left(\frac{1 + 0.8428}{2 - 0.8983} \right)^{1/4} \right]^3 = 1.47.$$

Since the Earth's current DTR (on land) is about 10 K, the DTR on an Earth-like planet without an atmosphere would be at most 14.7 K.

7.1. Affecting Changes in the DTR

We can now look at the effects on the DTR of changing ϵ , the emissivity/absorptivity of the atmosphere, versus changing Ω , the solar constant, related to how much sunlight is received by the planet. By taking the derivative of the DTR with respect to the equilibrium surface temperature with ϵ varying and Ω remaining constant and

then again with Ω varying and ϵ remaining constant, we can determine what effect various potential changes in the Earth's atmosphere and solar radiation will have on the global DTR.

First we will take the derivative of the DTR with respect to \hat{T}_0 while ϵ varies and Ω remains constant. In this scenario, using equation (63), we have:

$$\begin{aligned}
(66) \quad \frac{d}{d\hat{T}_0} \text{DTR} &= \frac{d(\text{DTR})}{d\omega} \frac{d\omega}{d\hat{T}_0} \\
&= \left(\frac{-2B}{\omega^2} \left(1 - \frac{2}{1+e^\omega} \right) + \frac{2B}{\omega} (f'(\omega)) \right) \frac{d\omega}{d\hat{T}_0} \\
&= \left(\frac{-2B}{\omega^2} (f(\omega)) + \frac{2B}{\omega} (f'(\omega)) \right) \frac{d\omega}{d\hat{T}_0} \\
&= 2B \left(\frac{-f(\omega)}{\omega^2} + \frac{f'(\omega)}{\omega} \right) \frac{d\omega}{d\hat{T}_0} \\
&= \frac{Pq\sigma T_e^4}{c_0} \left[\frac{-f(\omega)}{\omega^2} + \frac{f'(\omega)}{\omega} \right] \frac{3\omega}{\hat{T}_0} \\
&= \frac{3\omega}{\hat{T}_0} \frac{Pq\sigma T_e^4}{c_0} \left[\frac{-f(\omega)}{\omega^2} + \frac{f'(\omega)}{\omega} \right] \\
&= \frac{6P^2\sigma^2 q \hat{T}_0^2 T_e^4}{c_0^2} \left[\frac{-f(\omega)}{\omega^2} + \frac{f'(\omega)}{\omega} \right] \\
&= \frac{6P^2\sigma^2 q \hat{T}_0^6 T_e^4}{c_0^2 \hat{T}_0^4} \left[\frac{-f(\omega)}{\omega^2} + \frac{f'(\omega)}{\omega} \right] \\
&= 1.5q\omega^2 \left(\frac{T_e}{\hat{T}_0} \right)^4 \left[\frac{-f(\omega)}{\omega^2} + \frac{f'(\omega)}{\omega} \right] \\
&= 1.5 \frac{q}{[G(\epsilon, q)]^4} [-f(\omega) + \omega f'(\omega)] \\
&= \frac{q}{[G(\epsilon, q)]^4} \cdot 1.5 \left[\frac{2}{1+e^\omega} - 1 + \omega \left(\frac{2e^\omega}{(1+e^\omega)^2} \right) \right] \\
&= \frac{q}{[G(\epsilon, q)]^4} \cdot H_{\text{more gas}}(\omega),
\end{aligned}$$

where

$$H_{\text{more gas}}(\omega) = 3 \left[\frac{1}{1+e^\omega} - \frac{1}{2} + \frac{\omega e^\omega}{(1+e^\omega)^2} \right].$$

In order to make a similar calculation, but with Ω varying and ϵ remaining constant, we must first rewrite equation (63), which is an expression for the DTR

using (62) and (55):

$$\begin{aligned}
 (67) \quad \text{DTR} &= \frac{2B}{\omega} f(\omega) \\
 &= \frac{Pq\sigma T_e^4}{c_0} \cdot \frac{c_0}{2P\sigma\hat{T}_0^3} f(\omega) \\
 &= \frac{qT_e^4}{2\hat{T}_0^3} f(\omega) \\
 &= \frac{qT_e^4\hat{T}_0}{2\hat{T}_0^4} f(\omega) \\
 &= \frac{q\hat{T}_0 f(\omega)}{2[G(\epsilon, q)]^4}.
 \end{aligned}$$

We now take the derivative of the DTR with respect to \hat{T}_0 with Ω varying and ϵ constant:

$$\begin{aligned}
 (68) \quad \frac{d}{d\hat{T}_0} \text{DTR} &= \frac{q(f(\omega) + f'(\omega) \cdot 3\omega)}{2[G(\epsilon, q)]^4} \\
 &= \frac{q}{[G(\epsilon, q)]^4} \left[\frac{f(\omega) + f'(\omega) \cdot 3\omega}{2} \right] \\
 &= \frac{q}{[G(\epsilon, q)]^4} H_{\text{hotter sun}}(\omega),
 \end{aligned}$$

where

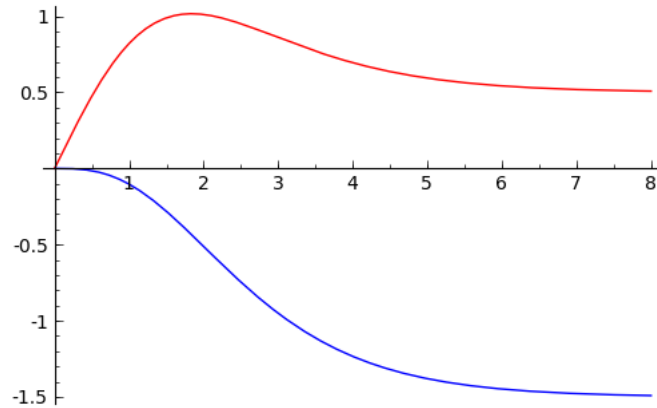
$$H_{\text{hotter sun}}(\omega) = \frac{1}{2} - \frac{1}{1 + e^\omega} + \frac{3\omega e^\omega}{(1 + e^\omega)^2}.$$

When the atmospheric emissivity and absorptivity, ϵ , is allowed to change, as ω increases, the DTR will decrease. When the temperature is increased by solar warming rather than by an increase in atmospheric density, the DTR will increase as ω increases (see Figure 7.2). In both cases, the H functions are multiplied by the term $\frac{q}{[G(\epsilon, q)]^4}$, which will give us a sense of the order of magnitude of the change of DTR. Using the values for the Earth, we get:

$$q[G(\epsilon, q)]^{-4} = 0.8428 \cdot [G(0.8983, 0.8428)]^{-4} = 0.5,$$

which means that somewhat significant changes in the DTR can occur on Earth when ω is large enough. You will notice in Figure 7.2, when ω is small, values of H

FIGURE 7.2. $H_{\text{more gas}}$ (blue) and $H_{\text{hotter sun}}$ (red). Note that $H_{\text{more gas}}$ is always negative and $H_{\text{hotter sun}}$ is always positive.



in both graphs are so small that they would lead to very inconsequential changes in the DTR.

LEMMA 20. For DTRs within the range $7 \leq \text{DTR} \leq 25$, ω is between 0.2 and 0.7, which means that the change in the DTR is:

- (1) between -0.0005 and -0.02 when ϵ varies, and
- (2) between 0.1 and 0.3 when Ω varies.

PROOF. DTRs on Earth tend to fall in the range between 7 and 25[7]. Using equation (67), we can solve the resulting inequality for ω as follows:

$$\begin{aligned}
7 &\leq \text{DTR} \leq 25 \\
7 &\leq \frac{q\hat{T}_0 f(\omega)}{2[G(\epsilon, q)]^4} \leq 25 \\
7 &\leq \frac{0.5}{2}\hat{T}_0 f(\omega) \leq 25 \\
28 &\leq \hat{T}_0 f(\omega) \leq 100 \\
28 &\leq 290f(\omega) \leq 100 \\
0.097 &\leq f(\omega) \leq 0.345 \\
0.097 &\leq 1 - \frac{2}{1 + e^\omega} \leq 0.345 \\
0.655 &\leq \frac{2}{1 + e^\omega} \leq 0.903 \\
0.3275 &\leq (1 + e^\omega)^{-1} \leq 0.4515 \\
2.215 &\leq 1 + e^\omega \leq 3.053 \\
1.215 &\leq e^\omega \leq 2.053 \\
0.2 &\leq \omega \leq 0.7.
\end{aligned}$$

If we consider ω to be within that range, when the atmosphere's emissivity is allowed to vary, we get a change in DTR within the range from -0.0005 to -0.02. When the solar constant is allowed to vary, we get a change in DTR within the range from 0.1 to 0.3. \square

This shows that, for the Earth, in our model the DTR is insensitive to changes in the emissivity, but it is much more sensitive to changes in the solar constant.

In reality, though, there has been an observed decrease in the DTR on Earth that many have associated with global climate change[4]. There are a few possible explanations for why this may be and why our model would not have picked up on these changes. One possible explanation is that the concentration of sulfate aerosols in the atmosphere has increased recently. Sulfate aerosols are microscopic particles that have entered the atmosphere due to the burning of fossil fuels (natural gas, oil, coal, etc.). Their presence in the atmosphere could have the effect of reflecting more

sunlight into space during the day. This would cause the warming caused by the greenhouse effect to slow down. This would not effect the greenhouse gasses effects on nighttime temperatures because there would be no incoming solar radiation for the sulfate aerosols to reflect at night, so the DTR would decrease with the days getting slightly colder and the nights remaining the same temperature.

CHAPTER 8

Conclusion

The model that we have created can provide us with some useful insight into climate change. We can see from our greenhouse function, equation (55), and our discussion on the absorptivity and emissivity of the atmosphere, ϵ , that CO₂ and other greenhouse gasses *do* have a warming effect on the equilibrium surface temperature. We can see from equations (66) and (68) that the change in the DTR as the surface temperature increases depends heavily on ω , which, according to equation (62) depends heavily on the surface temperature. Therefore it is plausible that given a significant increase in the surface temperature, which I mentioned has been predicted by more advanced models, our model could show a much more significant decrease in the DTR as a result of an increase in greenhouse gasses.

Although our simplified model does not accurately depict the total effect that climate change will have on the DTR, I have outlined in this paper various ways that we can go about building a more accurate model. Any future model that we may build will require the concepts discussed in this paper for its foundation.

Some areas for future research include looking further into the albedo. At the start of this paper we discussed the albedo briefly, but have thought of it as a constant value thus far. In fact, due to the albedo feedback effect, changes in the albedo can have a rather substantial effect on global climate that was not taken into account in this model. The albedo feedback effect is the process by which an initial increase in the global temperature causes melting of the icecaps, which reduces the albedo because the Earth becomes less reflective. This reduction in the albedo causes more solar radiation to be absorbed by the Earth and thus raises the global temperature. This process continues, causing a significant warming trend. A potential future model might take into account changes in the albedo as surface temperatures increase.

Another possibility for future research would be to look further into the effect that air and ocean circulation have on climate. In our model, we viewed each

hemisphere is an independent entity, but in reality, they have an effect on each other due to the circulation of the oceans and air currents. It would be interesting to see how this complicated system affects global climate.

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