New Results in Exponential Families

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Abstract

We start this paper by giving a short crash course in generating functions, including a new result that can be used to simplify certain operations. Then, in sections 2 and 3 we introduce H.S.Wilf's theory of exponential families (first introduced in his text *generatingfunctionology*), in section 4 we extend this theory by allowing restrictions on hand sizes. In particular, this allows us to expand Wilf's approach to treat the family of alternating groups A_n in the same way as Wilf treats the symmetric groups, allowing us to obtain the generating function for the cycle indicies of A_n.

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1 Short Introduction to Generating Functions

In this section we will introduce the notion of generating function, describe the two primary types of generating functions with examples, and provide tools on how to manipulate the resulting functions.

1.1 Ordinary Power Series Generating Functions

Definition 1.1. Given a sequence $\{a_n\}_{n=0}^{\infty}$, we say a function $f(x)$ is an *ordinary power series generating function* of $\{a_n\}$ if $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We indicate this by $\{a_n\} \overset{ops}{\longleftrightarrow} f(x)$, or $f(x) \overset{ops}{\longleftrightarrow} \{a_n\}$.

This is the most commonly encountered form of generating functions. For many common sequences, this function is useful because it is relatively easy to compute and admits an elegant closed form. In addition, it can often be used to compute a formula for the terms of the sequence from a recurrence relation for them.

Example 1.1. Consider the Fibonacci numbers. For our purposes, we define them as $f_0 = 0;\ f_1 = 1;\ f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$.

We can use Ordinary Power Series Generating Functions to compute the exact formula for these. To do so, we first find $F(x) \overset{ops}{\longleftrightarrow} \{f_n\}$.

$$\begin{aligned}
f_{n+2} &= f_{n+1} + f_n \\
f_{n+2}x^n &= f_{n+1}x^n + f_n x^n \\
\sum_{n=0}^{\infty} f_{n+2}x^n &= \sum_{n=0}^{\infty} f_{n+1}x^n + \sum_{n=0}^{\infty} f_n x^n \\
\frac{F(x) - x}{x^2} &= \frac{F(x)}{x} + F(x) \\
F(x) - x &= xF(x) + x^2F(x) \\
F(x) - xF(x) - x^2F(x) &= x \\
F(x)(1 - x - x^2) &= x \\
F(x) &= \frac{x}{1 - x - x^2}
\end{aligned}$$

Some of the steps here use methods described in section 1.3. Having obtained the generating function we can use partial fractions to compute an exact formula for the terms of the sequence. Let r_+ and r_- denote the positive and negative roots, respectively of $x^2 - x - 1 = 0$. Then

$$
\begin{aligned}
F(x) &= \frac{x}{1-x-x^2} \\
&= \frac{x}{(1-xr_+)(1-xr_-)} \\
&= \frac{1}{(r_+-r_-)}\left(\frac{1}{1-xr_+}-\frac{1}{1-xr_-}\right) \\
&= \frac{1}{\sqrt{5}}\left(\sum_{j=0}^{\infty} r_+^j x^j - \sum_{j=0}^{\infty} r_-^j x^j\right) \\
f_n = [x^n]\,F(x) &= \frac{1}{\sqrt{5}}\left(r_+^n - r_-^n\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)
\end{aligned}
$$

In the above, $[x^n]F(x)$ denotes the coefficient of x^n in $F(x)$.

1.2 Exponential Generating Functions

Definition 1.2. Given a sequence $\{a_n\}_{n=0}^{\infty}$, we say a function $f(x)$ is an *exponential generating function* of $\{a_n\}$ if $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$. We indicate this by $\{a_n\} \overset{egf}{\longleftrightarrow} f(x)$, or $f(x) \overset{egf}{\longleftrightarrow} \{a_n\}$.

These are sometimes used when the terms of the sequence grow quickly, since the $n!$ in the denominator may give a generating function with positive radius of convergence. They are often used in cases where $\{a_n\}$ are generated by a recurrence that involves a binomial coefficient (see table 1.3).

Example 1.2. We will now use the exponential generating function for the Bell numbers. The Bell numbers $\{b(n)\}$ may be defined by the relation $\{b(n)\} \overset{egf}{\longleftrightarrow} e^{e^x-1}$; in example 5.4, we establish that $b(n)$ is the number of partitions of an n-element set. We will put $B(x) := e^{e^x-1}$

$$
\begin{aligned}
B(x) - 1 = e^{e^x - 1} - 1 &= \left(\frac{1}{e}\right)\left(e^{e^x} - e\right) \\
&= \left(\frac{1}{e}\right)\sum_{r=0}^{\infty}\frac{1}{r!}(e^{rx} - 1) \\
&= \left(\frac{1}{e}\right)\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{n=1}^{\infty}\frac{(rx)^n}{n!} \\
&= \left(\frac{1}{e}\right)\sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{r=0}^{\infty}\frac{r^n}{r!} \\
&= \sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{r=0}^{\infty}\frac{1}{e}\frac{r^n}{r!}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
B(x) &= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{r=0}^{\infty}\frac{1}{e}\frac{r^n}{r!} \\
&= \sum_{n=0}^{\infty}\frac{x^n}{n!}b(n).
\end{aligned}
$$

From this we obtain the formula $b(n) = \frac{1}{e}\sum_{r=0}^{\infty}\frac{r^n}{r!}$.

1.3 Calculus of Generating Functions

It is often necessary to manipulate generating functions to obtain the generating functions for more complicated sequences, or, conversely, to manipulate sequences to find one generated by a given generating function. In this subsection we provide a number of very useful rules for such manipulation.

In the following table $A(x) \overset{ops}{\longleftrightarrow} \{a_n\},\ B(x) \overset{ops}{\longleftrightarrow} \{b_n\},\ \mathcal{A}(x) \overset{egf}{\longleftrightarrow} \{a_n\},$ and $\mathcal{B}(x) \overset{egf}{\longleftrightarrow} \{b_n\}$.

Sequence	OPSGF	EGF
1	$\frac{1}{1-x}$	e^x
$\alpha a_n + \beta b_n$	$\alpha A(x) + \beta B(x)$	$\alpha \mathcal{A}(x) + \beta \mathcal{B}(x)$
$n^k a_n$	$(xD)^k A(x)$	$(xD)^k \mathcal{A}(x)$
a_{n+k}	$\frac{A(x) - a_0 - \cdots - a_{k-1}x^{k-1}}{x^k}$	$D^k \mathcal{A}(x)$
a_{n-k}	$x^k A(x)$	
$\sum_{j=0}^n a_j b_{n-j}$	$A(x)B(x)$	
$\sum_{j=0}^n \binom{n}{j} a_j b_{n-j}$		$\mathcal{A}(x)\mathcal{B}(x)$

Table 1: Core Formulas of Calculus of Generating Functions

1.3.1 A Closer Look at the (xD) Operator

As we saw above the (xD) operator is used for both opsgf and egf. However, it can be very tedious to calculate $(xD)^k f(x)$, even when all the derivatives of f are well-known. However, it turns out that we can rearrange $(xD)^k$ into a linear combination of operators, each of which is of the form $x^j D^j$.

Theorem 1.1.

$$(xD)^n f(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k D^k f(x),$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ *is the Stirling number of the second kind, which satisfy the recurrence* $\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = 1$, $\left\{ \begin{array}{c} 0 \\ k \end{array} \right\} = 0$ *for* $k \neq 0$ *and* $\left\{ \begin{array}{c} n \\ k \end{array} \right\} = k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\}$ *for* $n > 0$.

Proof. We prove this theorem by induction on n.

Base case: Let $n = 0$. Since $\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = 1$, we obtain $(xD)^n f(x) = f(x) = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} x^0 D^0 f(x)$.

Inductive Step: Suppose the result has been shown for all $n < N$. Then we obtain:

$$
\begin{aligned}
(xD)^N f(x) &= (xD)\left((xD)^{N-1} f(x)\right) \\
&= (xD) \sum_{k=0}^{N-1} \left\{ \begin{array}{c} N-1 \\ k \end{array} \right\} x^k D^k f(x) \\
&= x \sum_{k=0}^{N-1} \left\{ \begin{array}{c} N-1 \\ k \end{array} \right\} \left(k x^{k-1} D^k f(x) + x^k D^{k+1} f(x) \right) \\
&= \sum_{k=0}^{N-1} \left\{ \begin{array}{c} N-1 \\ k \end{array} \right\} \left(k x^k D^k f(x) + x^{k+1} D^{k+1} f(x) \right) \\
&= \sum_{k=0}^{N} \left(k \left\{ \begin{array}{c} N-1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} N-1 \\ k-1 \end{array} \right\} \right) x^k D^k f(x) \\
&= \sum_{k=0}^{N} \left\{ \begin{array}{c} N \\ k \end{array} \right\} x^k D^k f(x)
\end{aligned}
$$

□

The significance of this result is that, together with a table of Stirling numbers (which needs to be calculated only once), it allows us to conveniently calculate the functions generating sequences of the form $n^k a_n$. This leads immediately to convenient calculation of $\sum P(n) a_n x^n$, where $P(n)$ is a polynomial and $\sum a_n x^n$ is available in closed form.

2 Exponential Families

In this section we examine the notion of an exponential family introduced by Wilf [**?**], and the rest of the paper will be focused on generalizing some of Wilf's results.

2.1 Definitions

In order to be able to study exponential families, we must first define what they are.

Definition 2.1. A *Card* $\mathcal{C}(S, p)$ is a "picture" p together with a finite set of positive integers (a "label set") S.

We also say a card $\mathcal{C}(S, p)$ has *weight* $n = |S|$.

Definition 2.2. A *relabeling* of a card $\mathcal{C}(S, p)$ is a new card $\mathcal{C}(S', p)$ with $|S'| = |S|$.

Definition 2.3. By a *hand* H we mean a collection of cards $\mathcal{C}_1(S_1, p_1), \ldots, \mathcal{C}_k(S_k, p_k)$, such that $\{S_1, \ldots, S_k\}$ is a partition of $[n]$.

We call n the *weight* of the hand.

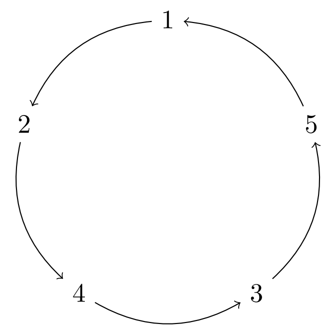

Figure 1: A sample card picture in $\mathbb{S}$

Definition 2.4. A *deck* $\mathcal{D}$ is a finite collection of cards such that the pictures are all distinct and all the label sets are $[n]$, for some fixed n.

n is called the *weight* of the deck.

Definition 2.5. An *exponential family* $\mathcal{F}$ is a collection of decks $\mathcal{D}_1, \mathcal{D}_2, \ldots$ such that the weight of deck $\mathcal{D}_n$ is n.

By the *picture set* of $\mathcal{F}$ we mean the set of all pictures from all cards in $\mathcal{F}$.

We say that a hand H *can be made in* $\mathcal{F}$ if each card in H is a relabeling of a card in a deck of $\mathcal{F}$.

Throughout this paper, we will return to the following example, which is useful for studying S_n.

Example 2.1. Each picture will be a circle, oriented counterclockwise, with the integers $1, \ldots, n$ arranged around it in any order (see Figure 1). Two pictures are identical if one can be rotated to match the other one.

To understand what a card is, we consider a set S of n positive integers. We then assign the smallest member of S to the location corresponding to 1 in p, the second smallest to 2, and so on. As a result, we have that a card of weight n is simply a cyclic permutation of S.

Then, for each n, the deck $\mathcal{D}_n$ in this exponential family is the set of all cyclic permutations of $[n]$, and so $\mathcal{D}_n$ must have $(n-1)!$ elements.

Finally, since each element of S_n can be uniquely decomposed into disjoint cycles (some might have length 1), the hands of weight n in this exponential family are exactly the elements of S_n. We will call this family $\mathbb{S}$.

The significance of the above example lies in the fact that it allows us to study all the S_n groups with a single structure. However, to be able to get useful information, we must first define the generating functions that we will use to study the exponential families.

Definition 2.6. Let $\mathcal{F}$ be an exponential family consisting of decks $\mathcal{D}_1, \mathcal{D}_2, \ldots$, then we let $d_r = |\mathcal{D}_r|$.

Let $\mathcal{D}(x) \overset{egf}{\longleftrightarrow} \{d_r\}_{r=1}^{\infty}$. We call $\mathcal{D}(x)$ the *deck enumerator* of $\mathcal{F}$.

Definition 2.7. Given an exponential family $\mathcal{F}$, let $h(n,k)$ denote the total number of hands of weight n consisting of k cards that can be made in this exponential family, by convention we let $h(0,0) = 1$, and let

$$\mathcal{H}(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h(n,k)\frac{x^n}{n!}y^k.$$

We call $\mathcal{H}(x,y)$ the *hand enumerator* of $\mathcal{F}$.

2.2 The Exponential Formula

Wilf's main result on exponential families is a closed formula for $\mathcal{H}(x,y)$. We include the proof in this section, because the proof of our main result is analogous in structure.

Theorem 2.1 (The Exponential Formula)**.** *Given any exponential family $\mathcal{F}$, with a hand enumerator $\mathcal{H}(x,y)$ and deck enumerator $\mathcal{D}(x)$,*

$$\mathcal{H}(x,y) = e^{y\mathcal{D}(x)}. \tag{1}$$

Before proving this theorem we must provide a method of manipulating exponential families. To do this, consider two exponential families $\mathcal{F}'$ and $\mathcal{F}''$, with disjoint picture sets P' and P'' respectively. Let the decks of $\mathcal{F}'$ be $\mathcal{D}_1', \mathcal{D}_2', \ldots$, and the decks of $\mathcal{F}''$ be $\mathcal{D}_1'', \mathcal{D}_2'', \ldots$. By $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ let us denote a new exponential family whose decks are $\mathcal{D}_1, \mathcal{D}_2, \ldots$, where $\mathcal{D}_n = \mathcal{D}_n' \cup \mathcal{D}_n''$.

Lemma 2.2 (The Fundamental Lemma of Labeled Counting)**.** *Let $\mathcal{F}'$ and $\mathcal{F}''$ be two exponential families with disjoint picture sets and hand enumerators $\mathcal{H}'(x,y)$ and $\mathcal{H}''(x,y)$ respectively; also let $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ have hand enumerator $\mathcal{H}(x,y)$. Then*

$$\mathcal{H}(x,y) = \mathcal{H}'(x,y)\mathcal{H}''(x,y).$$

Proof. Let $\mathcal{F}'$, $\mathcal{F}''$ and $\mathcal{F}$ be as described above, and let $h'(n,k)$, $h''(n,k)$ and $h(n,k)$ be the corresponding values as described in Definition 2.7.

Given a hand H in $\mathcal{F}$ with weight n and k cards, divide the cards of H into two subsets S' and S'' with cards from $\mathcal{F}'$ and $\mathcal{F}''$ respectively. This process generates a subset of $[n]$ (with n' elements, say) of integers that originally appeared on cards from $\mathcal{F}'$; a hand of weight n' with k' cards from $\mathcal{F}'$ and a hand of weight $n'' := n - n'$ with $k'' := k - k'$ cards from $\mathcal{F}''$. (The hands are constructed by order-preserving relabeling of S' with $[n']$ and of S'' with $[n'']$.)

Furthermore, this process generates an obvious bijection, giving us the following formula:

$$h(n,k) = \sum_{n'+n''=n} \sum_{k'+k''=k} \binom{n}{n'} h'(n',k')h''(n'',k'')$$
$$= \left[\frac{x^n}{n!}y^k\right] \mathcal{H}'(x,y)\mathcal{H}''(x,y).$$

The conclusion follows.

□

Using this lemma, we can build up the proof for the Exponential formula from simple building blocks.

Proof of The Exponential Formula. Let us first consider a very simple exponential family $\mathcal{F}_{r,1}$ which has only one non-empty deck, consisting of only a single card of weight r. So, $d_r = 1$ and $d_k = 0$ for $k \neq r$. Let the hand enumerator for $\mathcal{F}_{r,1}$ be $\mathcal{H}_{r,1}(x,y)$, and the deck enumerator be $\mathcal{D}_{r,1}(x)$. Then, $\mathcal{D}_{r,1}(x) = \dfrac{x^r}{r!}$. Also,

$$h(n,k) = \begin{cases} 0 & n \neq rk \\ \frac{(rk)!}{k!(r!)^k} & n = rk. \end{cases}$$

The second case above comes from the fact that we can choose the labels for the first card $\binom{n}{r}$ ways, for the second $\binom{n-r}{r}$, and so on, giving us $\dfrac{n!}{r!^k}$ ways to assign the labels, but the order of cards is immaterial, so we must divide by $k!$. The first case should be clear from the fact that each card must contribute exactly r to the weight of the hand.

From the above we have:

$$\mathcal{H}_{r,1}(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h(n,k)\frac{x^n}{n!}y^k$$
$$= \sum_{k=0}^{\infty} h(rk,k)\frac{x^{rk}}{(rk)!}y^k$$
$$= \sum_{k=0}^{\infty} \frac{(rk)!}{k!(r!)^k}\frac{x^{rk}}{(rk)!}y^k$$
$$= \sum_{k=0}^{\infty} \frac{\left(\frac{x^r y}{r!}\right)^k}{k!}$$
$$= e^{\frac{x^r y}{r!}} = e^{y\mathcal{D}_{r,1}(x)};$$

this establishes the theorem for this simple case.

Now consider $\mathcal{F}_r$ an exponential family with only one non-empty deck $\mathcal{D}_r$ containing d_r cards. Let $\mathcal{D}_r(x)$ and $\mathcal{H}_r(x,y)$ be the two enumerators for this family, with $\mathcal{D}_r(x) = \dfrac{d_r x^r}{r!} = d_r \mathcal{D}_{r,1}(x)$. We use the fundamental lemma to obtain the formula for $\mathcal{H}_r(x,y)$.

$$\begin{aligned}\mathcal{H}_r(x,y) = (\mathcal{H}_{r,1}(x,y))^{d_r} &= \left(e^{y\mathcal{D}_{r,1}(x)}\right)^{d_r} \\ &= e^{d_r y \mathcal{D}_{r,1}(x)} = e^{y\mathcal{D}_r(x)};\end{aligned}$$

this establishes the theorem for this case.

Finally, consider an arbitrary exponential family $\mathcal{F}$. Then $\mathcal{D}(x) = \sum\limits_{r=0}^{\infty} \frac{d_r x^r}{r!} = \sum\limits_{r=0}^{\infty} \mathcal{D}_r(x)$. The hand enumerator is $\mathcal{H}(x,y)$ which, using the lemma, we have as:

$$\begin{aligned}\mathcal{H}(x,y) = \prod_{r=1}^{\infty} \mathcal{H}_r(x,y) &= \prod_{r=1}^{\infty} e^{y\mathcal{D}_r(x)} \\ &= e^{y\sum_{r=1}^{\infty} \mathcal{D}_r(x)} = e^{y\mathcal{D}(x)};\end{aligned}$$

this completes the proof.

□

Let us now return to the family $\mathbb{S}$.

Example 2.2. We know that $d_r = (r-1)!$, so we have $\mathcal{D}(x) = \sum\limits_{r=1}^{\infty} \dfrac{x^r}{r} = \log \dfrac{1}{1-x}$.

So, applying the Exponential Formula, we get:

$$\mathcal{H}_{\mathbb{S}}(x,y) = e^{y\mathcal{D}(x)} = (1-x)^{-y}\,.$$

3 Exponential Families and Multiplicities

3.1 Introduction

While exponential families are powerful, they have a major shortcoming in that they do not "remember" what distribution of weights of individual cards in a hand were. To correct this, we will examine exponential families with multiplicities, which do preserve this information.

While we will still use the same definitions for exponential families, we must redefine the hand enumerator. But before doing this, let us introduce the following auxiliary notation.

Given a sequence of natural numbers $\overrightarrow{\alpha} = \langle \alpha_1, \alpha_2, \ldots \rangle$ that is eventually 0, we write $|\overrightarrow{\alpha}| = \sum_{r=1}^{\infty} \alpha_r$, and $\|\overrightarrow{\alpha}\| = \sum_{r=1}^{\infty} r\alpha_r$.

Let $\overrightarrow{y} = \langle y_1, y_2, \ldots \rangle$ be a sequence of variables, and $\overrightarrow{\alpha}$ be as above. We will write $\overrightarrow{y}^{\overrightarrow{\alpha}}$ or $\prod_{r=1}^{\infty} y_r^{\alpha_r}$ to mean $\prod_{r=1}^{N} y_r^{\alpha_r}$, where $\alpha_{N+1} = \alpha_{N+2} = \cdots = 0$.

Definition 3.1. Given $\overrightarrow{\alpha}$ as above and an exponential family $\mathcal{F}$, observe that any hand made with α_r cards from deck $\mathcal{D}_r$ $(r = 1, 2, \ldots)$ will have weight $n = \|\overrightarrow{\alpha}\|$. Let $\overrightarrow{h}(n, \overrightarrow{\alpha})$ be the number of such hands that can be made in this exponential family, as in Definition 2.7, we let $\overrightarrow{h}(0, \langle 0, 0, \ldots \rangle) = 1$. We refer to $\overrightarrow{\alpha}$ as the *multiplicity sequence* of such a hand.

Let

$$\overrightarrow{\mathcal{H}}(x, \overrightarrow{y}) = \sum_{\overrightarrow{\alpha}} \overrightarrow{h}(n, \overrightarrow{\alpha}) \frac{x^n}{n!} \overrightarrow{y}^{\overrightarrow{\alpha}}.$$

$\overrightarrow{\mathcal{H}}(x, \overrightarrow{y})$ will be called the *multiplicity hand enumerator* of $\mathcal{F}$.

A moment's reflection reveals that $\overrightarrow{\mathcal{H}}(x, \overrightarrow{y})$ evaluated at $y_1 = y_2 = \cdots = y$, is the hand enumerator $\mathcal{H}(x, y)$ defined in the previous section.

3.2 Exponential Formula With Multiplicities

The main result of this section is the closed form of $\overrightarrow{\mathcal{H}}(x, \overrightarrow{y})$ that is analogous to that of $\mathcal{H}(x, y)$ proved in Theorem 2.1. This result is exercise 22 on page 106 of [**?**].

Theorem 3.1. *The multiplicity hand enumerator of an exponential family $\mathcal{F}$ satisfies the identity*

$$\overrightarrow{\mathcal{H}}(x, \overrightarrow{y}) = \exp\left(\sum_{r=1}^{\infty} \frac{d_r y_r x^r}{r!}\right) \tag{2}$$

The proof will be analogous to that for Theorem 2.1. We begin with a multiplicity version of the fundamental lemma (Lemma 2.2).

Lemma 3.2. *Let $\mathcal{F}'$ and $\mathcal{F}''$ be two exponential families with disjoint picture sets and multiplicity hand enumerators $\overrightarrow{\mathcal{H}}'(x, \overrightarrow{y})$ and $\overrightarrow{\mathcal{H}}''(x, \overrightarrow{y})$ respectively; also let $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ have multiplicity hand enumerator $\overrightarrow{\mathcal{H}}(x, \overrightarrow{y})$. Then*

$$\overrightarrow{\mathcal{H}}(x, \overrightarrow{y}) = \overrightarrow{\mathcal{H}}'(x, \overrightarrow{y})\overrightarrow{\mathcal{H}}''(x, \overrightarrow{y}).$$

Proof. Let $\mathcal{F}'$, $\mathcal{F}''$ and $\mathcal{F}$ be as described above, and let $\overrightarrow{h}'(n, \overrightarrow{\alpha})$, $\overrightarrow{h}''(n, \overrightarrow{\alpha})$ and $\overrightarrow{h}(n, \overrightarrow{\alpha})$ be the corresponding values as described in Definition 3.1.

Given a hand H in $\mathcal{F}$ counted by $\overrightarrow{h}(n, \overrightarrow{\alpha})$, divide the cards of H into two subsets S' and S'' consisting of cards from $\mathcal{F}'$ and $\mathcal{F}''$ respectively. This process generates two multiplicity sequences $\overrightarrow{\alpha}'$ and $\overrightarrow{\alpha}''$ of cards in S' and S'' respectively; it also generates a subset of $[n]$ consisting of $n' = \|\overrightarrow{\alpha}'\|$ elements, namely those integers that appear on the cards of S'. As before, order-preserving relabelings of S' and S'' generate hands in $\mathcal{F}'$ and $\mathcal{F}''$ with multiplicity sequences $\overrightarrow{\alpha}'$ and $\overrightarrow{\alpha}''$.

Furthermore, this process generates an obvious bijection, giving us the following formula:

$$\begin{aligned}\overrightarrow{h}(n, \overrightarrow{\alpha}) &= \sum_{\overrightarrow{\alpha}' + \overrightarrow{\alpha}'' = \overrightarrow{\alpha}} \binom{n}{n'} \overrightarrow{h}'(n', \overrightarrow{\alpha}') \overrightarrow{h}''(n'', \overrightarrow{\alpha}'') \\ &= \left[\frac{x^n}{n!} \overrightarrow{y}^{\overrightarrow{\alpha}}\right] \overrightarrow{\mathcal{H}}'(x, \overrightarrow{y}) \overrightarrow{\mathcal{H}}''(x, \overrightarrow{y}),\end{aligned}$$

where $n' = \|\overrightarrow{\alpha}'\|$ and $n'' = \|\overrightarrow{\alpha}''\|$. The conclusion follows.

□

With this result established, we follow steps that correspond to those in the proof of the Theorem 2.1.

Proof of Theorem 3.1. Throughout this proof, the variable $\overrightarrow{\alpha}$ will range over multiplicity sequences, and we will put $n := \|\overrightarrow{\alpha}\|$.

Let us first consider a very simple exponential family $\mathcal{F}_{r,1}$ which has only one non-empty deck, consisting of only a single card of weight r. So, $d_r = 1$ and $d_k = 0$ for $k \neq r$. Let the multiplicity hand enumerator for $\mathcal{F}_{r,1}$ be $\overrightarrow{\mathcal{H}}_{r,1}(x, \overrightarrow{y})$.

First, observe that

$$\overrightarrow{h}(n, \overrightarrow{\alpha}) = \begin{cases} 0 & \alpha_j \neq 0 \ (\textit{for some } j \neq r) \\ \dfrac{(rk)!}{k!(r!)^k} & \alpha_r = k, \ \textit{and} \ \forall j \neq r, \ \alpha_j = 0. \end{cases}$$

The second case above comes from the fact that we can choose the labels for the first card $\binom{n}{r}$ ways, for the second $\binom{n-r}{r}$, and so on, giving us $\frac{n!}{r!^k}$ ways to assign the labels; however, since the order of cards is immaterial, we must divide by $k!$. The first case should be clear from the fact that we have no cards of weight other than r. We will use $\overrightarrow{\alpha}_{r,k}$ to denote the sequence $\overrightarrow{\alpha}$ where $\alpha_j = 0$ for $j \neq r$ and $\alpha_r = k$.

From the above we have:

$$
\begin{aligned}
\overrightarrow{\mathcal{H}}_{r,1}(x, \overrightarrow{y}) &= \sum_{\overrightarrow{\alpha}} \overrightarrow{h}(n, \overrightarrow{\alpha}) \frac{x^n}{n!} \overrightarrow{y}^{\overrightarrow{\alpha}} \\
&= \sum_{k=0}^{\infty} \overrightarrow{h}(rk, \overrightarrow{\alpha}_{r,k}) \frac{x^{rk}}{(rk)!} \overrightarrow{y}^{\overrightarrow{\alpha}_{r,k}} \\
&= \sum_{k=0}^{\infty} \frac{(rk)!}{k!(r!)^k} \frac{x^{rk}}{(rk)!} y_r^k \\
&= \sum_{k=0}^{\infty} \frac{\left(\frac{x^r y_r}{r!}\right)^k}{k!} \\
&= \exp\left(\frac{x^r y_r}{r!}\right).
\end{aligned}
$$

This establishes the theorem for this simple case.

Next, consider an exponential family $\mathcal{F}_r$ with only one non-empty deck $\mathcal{D}_r$ containing d_r cards. Let $\overrightarrow{\mathcal{H}}_r(x, \overrightarrow{y})$ be the multiplicity hand enumerator for this family. We use Lemma 3.2 to obtain the formula for $\overrightarrow{\mathcal{H}}_r(x, \overrightarrow{y})$:

$$
\begin{aligned}
\overrightarrow{\mathcal{H}}_r(x, \overrightarrow{y}) &= \left(\overrightarrow{\mathcal{H}}_{r,1}(x, \overrightarrow{y})\right)^{d_r} \\
&= \left(\exp\left(\frac{x^r y_r}{r!}\right)\right)^{d_r} \\
&= \exp\left(\frac{d_r x^r y_r}{r!}\right).
\end{aligned}
$$

This establishes the theorem for $\mathcal{F}_r$.

Finally, consider an arbitrary exponential family $\mathcal{F}$. Again using the lemma, we obtain a formula for $\overrightarrow{\mathcal{H}}(x, \overrightarrow{y})$, the multiplicity hand enumerator of $\mathcal{F}$:

$$
\begin{aligned}
\overrightarrow{\mathcal{H}}(x, \overrightarrow{y}) = \prod_{r=1}^{\infty} \overrightarrow{\mathcal{H}}_r(x, \overrightarrow{y}) &= \prod_{r=1}^{\infty} \exp\left(\frac{d_r x^r y_r}{r!}\right) \\
&= \exp\left(\sum_{r=1}^{\infty} \frac{d_r x^r y_r}{r!}\right).
\end{aligned}
$$

The proof is complete.

□

Example 3.1. Returning to the example from the previous section, we obtain the multiplicity hand enumerator for $\mathbb{S}$.

For this family, recall that $d_r = (r-1)!$. It follows that

$$\overrightarrow{\mathcal{H}}_{\mathbb{S}}(x, \overrightarrow{y}) = \exp\left(\sum_{r=1}^{\infty} \frac{x^r y_r}{r}\right).$$

4 Exponential Families with Restricted Hand Sizes

Theorem 3.1 not only provides a method for tracking multiplicities but also suggests the possibility of enumerating hands that satisfy multiplicity constraints. An interesting example is that of even permutations, which in $\mathbb{S}$ are realized as those hands in which the total number of even-weight cards is even. The main result of this paper establishes the means to derive the multiplicity hand enumerator of such exponential families.

Our first step is to restrict hand sizes; what we derive here is the multiplicity-tracking analogue of Corollary 3.4.1 on page 81 of [**?**]. We begin by introducing some new notation.

Let $R \subseteq \mathbb{N}$. Suppose that for some exponential family $\mathcal{F}$, we wish to allow only hands consisting of x cards, where $x \in R$. We call R the *hand size restriction* in $\mathcal{F}$. We define

$$\overrightarrow{h}_R(n, \overrightarrow{\alpha}) := \begin{cases} \overrightarrow{h}(n, \overrightarrow{\alpha}), & |\alpha| \in R \\ 0, & otherwise \end{cases}$$

and

$$\overrightarrow{\mathcal{H}}_R(x, \overrightarrow{y}) := \sum_{\overrightarrow{\alpha}} \overrightarrow{h}_R(n, \overrightarrow{\alpha}) \frac{x^n}{n!} \overrightarrow{y}^{\overrightarrow{\alpha}}.$$

We begin by deriving an explicit formula for $\overrightarrow{\mathcal{H}}_R$.

Theorem 4.1. *Given an exponential family $\mathcal{F}$ with hand size restriction R,*

$$\overrightarrow{\mathcal{H}}_R(x, \overrightarrow{y}) = \exp_R\left(\sum_{r=1}^{\infty} \frac{d_r x^r y_r}{r!}\right),$$

where $\exp_R(z) = \sum_{k \in R} \frac{z^k}{k!}$.

Proof. First observe that

$$\sum_{k=0}^{\infty} \sum_{|\vec{\alpha}|=k} \vec{h}(n, \vec{\alpha}) \frac{x^n}{n!} \vec{y}^{\vec{\alpha}} = \sum_{\vec{\alpha}} \vec{h}(n, \vec{\alpha}) \frac{x^n}{n!} \vec{y}^{\vec{\alpha}}$$
$$= \vec{\mathcal{H}}(x, \vec{y})$$
$$= \exp\left(\sum_{r=1}^{\infty} \frac{d_r x^r y_r}{r!} \right) = \sum_{k=0}^{\infty} \frac{\left(\sum_{r=1}^{\infty} \frac{d_r x^r y_r}{r!} \right)^k}{k!}.$$

Next, since the first and last expressions above both partition the sum by the total exponent of $\vec{y}$, the separate summands are equal:

$$\sum_{|\vec{\alpha}|=k} \vec{h}(n, \vec{\alpha}) \frac{x^n}{n!} \vec{y}^{\vec{\alpha}} = \frac{\left(\sum_{r=1}^{\infty} \frac{d_r x^r y_r}{r!} \right)^k}{k!}.$$

Finally, summing over $k \in R$, we get

$$\vec{\mathcal{H}}_R(x, \vec{y}) = \sum_{k \in R} \frac{\left(\sum_{r=1}^{\infty} \frac{d_r x^r y_r}{r!} \right)^k}{k!}.$$

□

This theorem enables us, for example, to find the hand enumerator for the family of permutations with an even number of cycles. However, it is not powerful enough to give us the hand enumerator for $\{A_n\}$; to remedy this, we must allow separate restrictions for different subsets of cards. Given exponential families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ with disjoint picture sets and hand size restrictions $R_1, R_2, \ldots, R_m$ respectively, let $W = R_1 \times R_2 \times \cdots \times R_m \subseteq \mathbb{N}^m$, and let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_m$.

Given a hand in $\mathcal{F}$ with multiplicity sequence $\vec{\alpha}$, let $\vec{\alpha}^{(j)}$ $(1 \leq j \leq m)$ be the multiplicity sequence of the cards from $\mathcal{F}_j$, and let $k_j := |\vec{\alpha}^{(j)}|$ be the number of cards from $\mathcal{F}_j$. We now define $\vec{h}_W(n, \vec{\alpha})$ to be the number of hands with multiplicity sequence $\vec{\alpha}$ such that $(k_1, k_2, \ldots, k_m) \in W$. We also define

$$\vec{\mathcal{H}}_W(x, \vec{y}) := \sum_{\vec{\alpha}} \vec{h}_W(n, \vec{\alpha}) \frac{x^n}{n!} \vec{y}^{\vec{\alpha}}.$$

Theorem 4.2 (Main Theorem). *Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$, $R_1, R_2, \ldots, R_m$, W, $\mathcal{F}$, and $\vec{\mathcal{H}}_W(x, \vec{y})$ be as above. Then,*

$$\vec{\mathcal{H}}_W(x, \vec{y}) = \prod_{j=1}^{m} \vec{\mathcal{H}}_{j,R_j}(x, \vec{y}),$$

where $\overrightarrow{\mathcal{H}}_{j,R_j}(x,\overrightarrow{y})$ is the multiplicity hand enumerator of $\mathcal{F}_j$ with hand size restriction R_j.

Lemma 4.3.

$$\overrightarrow{h}_W(n,\overrightarrow{\alpha}) = \sum_{\overrightarrow{\alpha}^{(1)}+\cdots+\overrightarrow{\alpha}^{(m)}=\overrightarrow{\alpha}} \left(\frac{n!}{\|\overrightarrow{\alpha}^{(1)}\|!\cdots\|\overrightarrow{\alpha}^{(m)}\|!} \prod_{j=1}^{m} \overrightarrow{h}_{j,R_j}(\|\overrightarrow{\alpha}^{(j)}\|,\overrightarrow{\alpha}^{(j)}) \right).$$

Proof. Given a hand H counted by $\overrightarrow{h}_W(n,\overrightarrow{\alpha})$, divide the cards into subsets $S_1,\ldots,S_m$ consisting of cards from $\mathcal{F}_1,\ldots,\mathcal{F}_m$ respectively. This process generates $\overrightarrow{\alpha}^{(1)},\ldots,\overrightarrow{\alpha}^{(m)}$, the multiplicity sequences of the respective subsets of cards. Furthermore, since $\left(\left|\overrightarrow{\alpha}^{(1)}\right|,\ldots,\left|\overrightarrow{\alpha}^{(m)}\right|\right)\in W$, for each j we have $\left|\overrightarrow{\alpha}^{(j)}\right|\in R_j$, so that order-preserving relabeling gives a hand with multiplicity sequence $\overrightarrow{\alpha}^{(j)}$ that is counted by $\overrightarrow{h}_{j,R_j}(\|\overrightarrow{\alpha}^{(j)}\|,\overrightarrow{\alpha}^{(j)})$. This process also generates an ordered partition of $[n]$ into m (potentially empty) subsets, namely the labels of cards in $S_1,\ldots,S_m$. Since this process is bijective, the result follows. □

Proof of the Main Theorem. Let us start with the right hand side of the equation:

$$\begin{aligned}
\prod_{j=1}^{m}\overrightarrow{\mathcal{H}}_{j,R_j}(x,\overrightarrow{y}) &= \prod_{j=1}^{m}\sum_{\overrightarrow{\alpha}^{(j)}} \overrightarrow{h}_{j,R_j}(\|\overrightarrow{\alpha}^{(j)}\|,\overrightarrow{\alpha}^{(j)}) \frac{x^{\|\overrightarrow{\alpha}^{(j)}\|}}{\|\overrightarrow{\alpha}^{(j)}\|!}\overrightarrow{y}^{\overrightarrow{\alpha}^{(j)}} \\
&= \sum_{\overrightarrow{\alpha}} \left(\sum_{\overrightarrow{\alpha}^{(1)}+\cdots+\overrightarrow{\alpha}^{(m)}=\overrightarrow{\alpha}} \left(\frac{n!}{\|\overrightarrow{\alpha}^{(1)}\|!\cdots\|\overrightarrow{\alpha}^{(m)}\|!} \prod_{j=1}^{m} \overrightarrow{h}_{j,R_j}(\|\overrightarrow{\alpha}^{(j)}\|,\overrightarrow{\alpha}^{(j)}) \right)\right) \frac{x^n}{n!}\overrightarrow{y}^{\overrightarrow{\alpha}} \\
&= \sum_{\overrightarrow{\alpha}} \overrightarrow{h}_W(n,\overrightarrow{\alpha})\frac{x^n}{n!}\overrightarrow{y}^{\overrightarrow{\alpha}} \\
&= \overrightarrow{\mathcal{H}}_W(x,\overrightarrow{y}).
\end{aligned}$$

This establishes our main theorem.

□

Remark. This result can be extended to infinitely many families given the following restrictions:

1. $0\in R_j$ for all but finitely many values of j.
2. Each deck $\mathcal{D}_r$ is empty except in finitely many families.

Note. A possible direction of further study would be to examine what occurs if we let W be an arbitrary subset of $\mathbb{N}^m$.

5 Examples of Exponential Families with Restricted Hand Sizes

We end this paper with a collection of results that can be obtained as applications of Theorem 4.1 and the Main Theorem.

5.1 Subsets of S_n

We begin with several examples obtained from interesting subsets of S_n. Later we will consider the question of which subsets can be treated by this method.

Example 5.1. Consider the alternating groups A_n. A permutation is an element of A_n iff it has an even number of even-length cycles. Let us call a multiplicity sequence $\overrightarrow{\alpha}$ even if $\sum_{j\ even} \alpha_j$ is even. Then the multiplicity hand enumerator for $\{A_n\}$ is

$$\overrightarrow{\mathcal{H}}_{\mathbb{A}}(x, \overrightarrow{y}) := \sum_{\overrightarrow{\alpha}\ even} \overrightarrow{h}(n, \overrightarrow{\alpha}) \frac{x^n}{n!} \overrightarrow{y}^{\overrightarrow{\alpha}}$$

where $\overrightarrow{h}(n, \overrightarrow{\alpha})$ is the same as in Example 3.1.

We now find the closed form for $\overrightarrow{\mathcal{H}}_{\mathbb{A}}(x, \overrightarrow{y})$.

Let $\mathcal{F}_1$ be the exponential family with odd decks from $\mathbb{S}$ and even decks empty, with $R_1 = \mathbb{N}$.

Let $\mathcal{F}_2$ be the exponential family with even decks from $\mathbb{S}$ and odd decks empty, with $R_2 = 2\mathbb{N}$.

We use $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, and let $W = R_1 \times R_2$, so that $\overrightarrow{\mathcal{H}}_{\mathbb{A}}(x, \overrightarrow{y}) = \overrightarrow{\mathcal{H}}_W(x, \overrightarrow{y})$.

By Theorem 3.1

$$\overrightarrow{\mathcal{H}}_{R_1}(x, \overrightarrow{y}) = \exp\left(\sum_{r\ odd} \frac{x^r y_r}{r}\right),$$

and by Theorem 4.1,

$$\overrightarrow{\mathcal{H}}_{R_2}(x, \overrightarrow{y}) = \sum_{k\ even} \frac{\left(\sum_{r\ even} \frac{x^r y_r}{r}\right)^k}{k!} = \cosh\left(\sum_{r\ even} \frac{x^r y_r}{r}\right).$$

Finally, by the Main Theorem

$$\begin{aligned}
\overrightarrow{\mathcal{H}}_{\mathbb{A}}(x, \overrightarrow{y}) &= \overrightarrow{\mathcal{H}}_W(x, \overrightarrow{y}) = \overrightarrow{\mathcal{H}}_{R_1}(x, \overrightarrow{y})\overrightarrow{\mathcal{H}}_{R_2}(x, \overrightarrow{y}) \\
&= \exp\left(\sum_{r\ odd} \frac{x^r y_r}{r}\right) \cosh\left(\sum_{r\ even} \frac{x^r y_r}{r}\right) \\
&= \exp\left(\sum_{r\ odd} \frac{x^r y_r}{r}\right) \frac{1}{2}\left[\exp\left(\sum_{r\ even} \frac{x^r y_r}{r}\right) + \exp\left(-\sum_{r\ even} \frac{x^r y_r}{r}\right)\right] \\
&= \frac{1}{2}\left[\exp\left(\sum_{r=1}^{\infty} \frac{x^r y_r}{r}\right) + \exp\left(\sum_{r=1}^{\infty} \frac{(-x)^r(-y_r)}{r}\right)\right] \\
&= \frac{1}{2}\left[\overrightarrow{\mathcal{H}}_{\mathbb{S}}(x, \overrightarrow{y}) + \overrightarrow{\mathcal{H}}_{\mathbb{S}}(-x, -\overrightarrow{y})\right].
\end{aligned}$$

We will return to this formula at the end of this subsection.

It is also interesting to consider the hand enumerator for $\{A_n\}$ without multiplicities (obtained by equating $y_1 = y_2 = \cdots = y$). Since $\exp\left(y \sum \frac{x^r}{r}\right) = (1-x)^{-y}$ (see example 2.2), this hand enumerator simplifies to:

$$\sum_{n,k} (\#\text{ of elements of } A_n \text{ with } k \text{ cycles}) \frac{x^n}{n!} y^k = \frac{1}{2}\left[(1-x)^{-y} + (1+x)^y\right].$$

Let us now introduce some notation that will be used in our next example:

For p and n positive integers with $p > 1$, $e(p, n)$ will denote the highest power of p that divides n.

We put $((m, k)) := \prod_{p|m} p^{e(p,k)}$, where the product ranges over prime divisors of m.

We will also write $\exp_k(x) := \exp_{k\mathbb{N}}(x) = \sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!}$, where k is a positive integer.

Example 5.2. Consider the family of permutations that have k^{th} roots in S_n.

In [**?**], Wilf proves that a permutation has a k^{th} root iff the number of m cycles it has is divisible by $((m, k))$.

For all positive integers m, let $\mathcal{F}_m$ be the exponential family that has only one non-empty deck ($\mathcal{D}_m$), which is identical to $\mathcal{D}_m$ in $\mathbb{S}$, and let R_m be all non-negative integers divisible by $((m, k))$.

Then given $W = R_1 \times R_2 \times \cdots$, we know that $\overrightarrow{\mathcal{H}}_W(x, \overrightarrow{y})$ counts the number of permutations that have a k^{th} root.

$$\begin{aligned}
\overrightarrow{\mathcal{H}}_W(x, \overrightarrow{y}) &= \prod_{m=1}^{\infty} \overrightarrow{\mathcal{H}}_{m,R_m}(x, \overrightarrow{y}) \\
&= \prod_{m=1}^{\infty} \exp_{((m,k))}\left(\frac{x^m y_m}{m}\right).
\end{aligned}$$

Example 5.3. If we set $k = 2$, the previous example becomes much more concrete: we are interested in counting the permutations that have square roots. Also,

$$((m, 2)) = \begin{cases} 1, & m \text{ odd} \\ 2, & m \text{ even} \end{cases}$$

In which case,

$$\begin{aligned} \overrightarrow{\mathcal{H}}_W(x, \overrightarrow{y}) &= \prod_{m\ odd} \exp\left(\frac{x^m y_m}{m}\right) \times \prod_{m\ even} \cosh\left(\frac{x^m y_m}{m}\right) \\ &= \exp\left(\sum_{m\ odd} \frac{x^m y_m}{m}\right) \times \prod_{m\ even} \cosh\left(\frac{x^m y_m}{m}\right) \end{aligned}$$

Remark. A finite group G acting on a finite set X generates a polynomial of combinatorial interest called the *cycle index of* G; when G is a subgroup of S_n (thought of as acting on $[n]$), the cycle index is

$$P_G(y_1, \ldots, y_n) := \frac{1}{|G|} \sum_{g \in G} y_1^{b_1} \cdots y_n^{b_n},$$

where g has *cycle type* $(b_1, \ldots, b_n)$; that is, g has b_1 fixed points, b_2 2-cycles, etc.

In the case $G = S_n$, Wilf [**?**] and DeBruijn [**?**], by completely different methods, establish that the opsgf with coefficients $P_{S_n}(y_1, \ldots, y_n)$ is:

$$\sum_{n=0}^{\infty} P_{S_n}(y_1, \ldots, y_n) x^n = \overrightarrow{\mathcal{H}}_{\mathbb{S}}(x, \overrightarrow{y}).$$

Building on example 5.1, we can find the closed form for the opsgf with coefficients $P_{A_n}(y_1, \ldots, y_n)$:

$$\sum_{n=0}^{\infty} P_{A_n}(y_1, \ldots, y_n) x^n = 2\overrightarrow{\mathcal{H}}_{\mathbb{A}}(x, \overrightarrow{y}) = \overrightarrow{\mathcal{H}}_{\mathbb{S}}(x, \overrightarrow{y}) + \overrightarrow{\mathcal{H}}_{\mathbb{S}}(-x, -\overrightarrow{y}).$$

A question of interest is what other subgroups of S_n can be treated by our methods. It turns out that if we don't allow partial decks, so that the membership or otherwise of a permutation σ in a subgroup G is determined by the cycle type of σ, then (for $n \neq 4$), the only groups that yield to our methods are S_n, A_n and $\{e\}$. This is because all such groups G must be normal in S_n, and it is easy to show that there are only 3 normal subgroups of S_n, namely, S_n, A_n and $\{e\}$.

Note. A further study of applications of exponential families to subsets of S_n could focus on classifying which subsets can be treated when we remove the restriction of using entire decks.

5.2 Partitions of $[n]$

Another class of examples that our approach can be applied to are partitions of $[n]$.

Example 5.4. We start with the description of the family for this class of examples, as well as the hand enumerator for the entire family.

Consider the family $\mathbb{P}$ where the pictures on each card are just $[r]$, for some $r \geq 1$, with labeling accomplished analogously to $\mathbb{S}$. Then, a card can be thought of as just its label set. Therefore, each deck $\mathcal{D}_r$ consists of a single card (namely $[r]$), so $\mathcal{D}(x) = e^x - 1$.

We get that $h(n,k)$ is the number of partitions of $[n]$ into k subsets. However, this is just the definition of the Stirling numbers of the second kind, which gives us the following for $\mathcal{H}_{\mathbb{P}}(x,y)$:

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left\{ \begin{array}{c} n \\ k \end{array} \right\}\frac{x^n}{n!}y^k &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}h(n,k)\frac{x^n}{n!}y^k = \mathcal{H}_{\mathbb{P}}(x,y) \\ &= \exp\left(y(e^x-1)\right).\end{aligned}$$

It is interesting to note that by evaluating at $y=1$ we obtain:

$$\sum_{n=0}^{\infty} b(n)\frac{x^n}{n!} = e^{e^x-1} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\left\{ \begin{array}{c} n \\ k \end{array} \right\}\right)\frac{x^n}{n!},$$

where $b(n) = \sum_{k=0}^{\infty}\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the number of partitions of an n-element set.

As we have already seen in Example 1.2, this generating function can be used to compute a formula for the Bell numbers.

We can also calculate the multiplicity hand enumerator. Simply applying Theorem 3.1 we get:

$$\overrightarrow{\mathcal{H}}_{\mathbb{P}}(x,\overrightarrow{y}) = \exp\left(\sum_{r=1}^{\infty}\frac{x^r y_r}{r!}\right).$$

Example 5.5. The other example with $\mathbb{P}$ that we consider is the one where we limit the partitions to have at most m sets of any given size.

To achieve this, we simply apply Theorem 4.1, with $R = \{0,1,2,\ldots,m\}$:

$$\overrightarrow{\mathcal{H}}_{\mathbb{P},R}(x,\overrightarrow{y}) = \sum_{k=0}^{m}\frac{\left(\sum_{r=1}^{\infty}\frac{x^r y_r}{r!}\right)^k}{k!}.$$

There are a multitude of other examples based on the exponential family $\mathbb{P}$, just as there are for $\mathbb{S}$.

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